

Algorithms for Trigonometric Wavelet Packets

EWALD QUAK¹

SINTEF Applied Mathematics, P.O. Box 124 Blindern, N-0314 Oslo, Norway

AND

NORMAN WEYRICH²

Synopsys GmbH, Kaiserstrasse 100, D-52134 Herzogenrath, Germany

Communicated by M. V. Wickerhauser

Received January 23, 1995; revised October 8, 1996

The aim of this paper is to describe wavelet packet functions and spaces for a trigonometric multiresolution analysis based on fundamental Lagrange interpolants. The corresponding algorithms for wavelet packet decomposition and reconstruction are investigated in detail. As an example, an application of trigonometric wavelet packets to speech processing is considered. © 1997

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1. INTRODUCTION

Trigonometric Lagrange interpolants have been widely studied in approximation theory, see for example [33]. The particular functions used in this paper were studied by Privalov [23] in the search for orthogonal trigonometric bases of optimal degree for the space of continuous 2π -periodic functions. This problem was further investigated in [18] and eventually completely resolved in [16] using wavelet and wavelet packet techniques, respectively. The first approach to develop a trigonometric multiresolution analysis of nested spaces of trigonometric polynomials was introduced by Chui & Mhaskar [2], with their investigations being based on quasi-interpolants. Various aspects of MRA's such as decomposition and reconstruction algorithms, dual and orthogonal bases, localization properties, etc., derived from Privalov's Lagrange interpolants, were studied by Prestin, Privalov, Quak, and Selig in [19–24, 26].

In this paper, trigonometric wavelet packet functions are investigated and the corresponding algorithms are formulated and discussed. The concept of wavelet packets for further analysing the wavelet components of a wavelet de-

composition was introduced by Coifman *et al.* [6]. This study, along with the so-called best bases algorithms for wavelet packets, has led to considerable research activity and applications in speech processing and other areas (see [3–5, 7–11, 15, 29–32]).

The paper is organized as follows. After reviewing the necessary basic properties in Section 2, we introduce in Section 3 the matrix representations of ordinary trigonometric wavelet decomposition and reconstruction using fast Fourier transformation (FFT) techniques and formulate the respective algorithms. In Section 4, the actual wavelet packet functions are described along with their basic interpolatory properties. The corresponding wavelet packet spaces are defined and results are presented concerning the question which trigonometric polynomials can be found in a given packet space and—conversely—how to determine the packet space that contains a given polynomial. Section 5 states the matrix representation for packet decomposition and reconstruction, again with a detailed description of the FFT based algorithms. While the results of the earlier sections pertain to interpolatory semi-orthogonal trigonometric wavelets, Section 6 contains the necessary modifications so orthogonal (non-interpolatory) wavelet packets can be generated. The final Section 7 on applications first gives operation counts and computation time comparisons to packets based on other wavelet types, then concludes with an application in speech processing.

2. BASIC PROPERTIES OF A TRIGONOMETRIC MRA

For $\ell \in \mathbb{N}$, the Dirichlet kernel $D_\ell \in T_\ell$ is defined as

$$D_\ell(x) = \frac{1}{2} + \sum_{k=1}^{\ell} \cos kx = \begin{cases} \frac{\sin(\ell + \frac{1}{2})x}{2 \sin \frac{x}{2}} & \text{for } x \notin 2\pi\mathbb{Z}, \\ \ell + \frac{1}{2} & \text{for } x \in 2\pi\mathbb{Z}, \end{cases} \quad (2.1)$$

¹E-mail: Ewald.Quak@math.sintef.no

²E-mail: weyrich@synopsys.com

where T_ℓ denotes the linear space of trigonometric polynomials of degree ℓ .

In the following, a slight modification of such a simple Dirichlet kernel is used to construct certain interpolatory operators. Namely, let

$$\begin{aligned}\phi_{j,0}(x) &= \frac{1}{2^{j+1}} \sum_{\ell=2^j-1}^{2^j} D_\ell(x) \\ &= \begin{cases} \frac{\sin(2^j x) \cot \frac{x}{2}}{2^{j+1}} & \text{for } x \notin 2\pi\mathbb{Z}, \\ 1 & \text{for } x \in 2\pi\mathbb{Z}. \end{cases} \quad (2.2)\end{aligned}$$

Such a simple sum is the special case $N = 2^j$ and $M = 1$ of very general de la Vallée–Poussin kernels

$$\phi_M^N(x) = \frac{1}{2NM} \sum_{\ell=N-M}^{N+M-1} D_\ell(x)$$

used for trigonometric wavelet decompositions [19–22, 26]. For $M > 1$, the distribution of sine and cosine frequencies in the corresponding scaling function and wavelet spaces is more complicated (see Theorem 7.1 in [19] and Theorem 3.4 in [22]). For simplicity, this paper is restricted to the case (2.2), but all results can and will be adapted to the more general kernels in due course.

The crucial interpolatory property of $\phi_{j,0}$ is

$$\phi_{j,0}\left(\frac{k\pi}{2^j}\right) = \delta_{k,0}, \quad k = 0, 1, \dots, 2^{j+1} - 1. \quad (2.3)$$

DEFINITION 2.1. For $j \in \mathbb{N}_0$, the spaces V_j are defined by $V_j = \text{span}\{\phi_{j,n} : n = 0, \dots, 2^{j+1} - 1\}$, where $\phi_{j,n}(x) = \phi_{j,0}(x - n\pi/2^j)$.

For notational convenience, let $\phi_{j,n} = \phi_{j,n \bmod 2^{j+1}}$ for any $n \in \mathbb{Z}$. The dimension of the spaces V_j clearly is 2^{j+1} due to the corresponding interpolatory property

$$\phi_{j,n}\left(\frac{k\pi}{2^j}\right) = \delta_{k,n}, \quad k, n \in \mathbb{Z},$$

which also shows that these functions are in fact fundamental functions for Lagrange interpolation.

One can show that, for all $j \in \mathbb{N}_0$, it holds that

$$V_j = \text{span}\{1, \cos x, \dots, \cos(2^j - 1)x, \sin x, \dots, \sin(2^j - 1)x\}.$$

Therefore, $V_j \subset V_{j+1}$, i.e., the spaces V_j form a sequence of nested subspaces of L^2 , the space of 2π -periodic square integrable functions. Setting $V_{-1} = \{0\}$, it is also clear that

$$L^2 = \text{clos}_{L^2} \left(\bigcup_{j=-1}^{\infty} V_j \right) \quad \text{and} \quad \bigcap_{j=-1}^{\infty} V_j = \{0\}.$$

As the next step, the orthogonal complement of V_j relative to V_{j+1} , i.e., the so-called wavelet space W_j is described in more detail.

DEFINITION 2.2. For $j \in \mathbb{N}_0$, the spaces W_j are defined by $W_j = \text{span}\{\psi_{j,n} : n = 0, \dots, 2^{j+1} - 1\}$, where

$$\psi_{j,n}(x) = 2\phi_{j+1,2n+1}(x) - \phi_{j,n}\left(x - \frac{\pi}{2^{j+1}}\right) \in V_{j+1}. \quad (2.4)$$

The functions $\psi_{j,n}$ also show interpolatory properties, namely for all $k \in \mathbb{Z}$,

$$\begin{aligned}\psi_{j,n}\left(\frac{(2k+1)\pi}{2^{j+1}}\right) &= \delta_{k,n} \quad \text{and} \\ \psi_{j,n}\left(\frac{k\pi}{2^j}\right) &= -\phi_{j,n}\left(\frac{(2k-1)\pi}{2^{j+1}}\right).\end{aligned} \quad (2.5)$$

Therefore, $\dim W_j = 2^{j+1}$.

Let $\langle \cdot, \cdot \rangle$ denote the inner product of two functions f and g in L^2 , i.e.,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx.$$

Then the following theorem can be established.

THEOREM 2.1 [23]. *The spaces V_j and W_j are orthogonal, i.e.,*

$$\langle \phi_{j,n}, \psi_{j,k} \rangle = 0 \quad \text{for all } n, k \in \mathbb{Z}.$$

Consequently, with \oplus denoting orthogonal summation,

$$V_{j+1} = V_j \oplus W_j.$$

A relation between V_{j+1} and V_j based solely on dilation is incompatible with the functions $\phi_{j,k}$ being periodic fundamental interpolants. Still, the following can be computed directly using the definitions of $\phi_{j,0}$ and $\psi_{j,2^{j+1}-1}$

- (i) $\phi_{j+1,0}(x) = \phi_{j,0}(2x) \left(\frac{1}{2} + \frac{1}{2\cos x} \right),$
- (ii) $\psi_{j+1,2^{j+2}-1}(x) = \psi_{j,2^{j+1}-1}(2x) \left(\frac{1}{2} + \frac{1}{2\cos x} \right).$

Note that the corrective factor $1/2 + 1/(2\cos x)$ is independent of the level j .

In this paper, ample use will be made of matrix representations to describe decomposition and reconstruction relations, orthogonality properties, etc. The main reason for this lies in the convenient fact that due to the periodicity involved, all relevant matrices are circulant and all such circulants can be factorized as the product of a so-called Fourier matrix, a diagonal matrix containing its eigenvalues and the conjugate (i.e., inverse) of the Fourier matrix. For more information concerning the notation and all necessary details, see [13], especially Theorem 3.2.4. In our setting, let a circulant matrix A_{j+1} of dimension 2^{j+1} be given by its first column, i.e.,

$$A_{j+1} = \text{circ}(a_{j+1,(0,0)}, \dots, a_{j+1,(2^{j+1}-1,0)}),$$

then it can be factorized as

$$A_{j+1} = \bar{F}_{j+1} \alpha_{j+1} F_{j+1},$$

where

$$F_{j+1} := \frac{1}{\sqrt{2^{j+1}}} \left(\exp \left(\frac{rs\pi i}{2^j} \right) \right)_{r,s=0}^{2^{j+1}-1} \quad \text{and}$$

$$\alpha_{j+1} := \text{diag} \left(\sum_{k=0}^{2^{j+1}-1} a_{j+1,k,0} \exp \left(\frac{kr\pi i}{2^j} \right) \right)_{r=0}^{2^{j+1}-1}.$$

Thus, defining the symmetric circulant inner product matrix G_{j+1} for the scaling functions $\phi_{j,r}$ as

$$G_{j+1} = (\langle \phi_{j,r}, \phi_{j,s} \rangle)_{r,s=0,\dots,2^{j+1}-1},$$

we have

THEOREM 2.2 [22]. *For any $j \in \mathbb{N}_0$, the inner product matrix for the scaling functions can be factorized as*

$$G_{j+1} = \bar{F}_{j+1} \Gamma_{j+1} F_{j+1},$$

with $\Gamma_{j+1} = \text{diag}(\gamma_{j+1,r})_{r=0,\dots,2^{j+1}-1}$, where the eigenvalues are

$$\gamma_{j+1,r} = \begin{cases} \frac{1}{2^{j+1}} & \text{for } 0 \leq r \leq 2^j - 1, \\ \frac{1}{2^{j+2}} & \text{for } r = 2^j, \\ \frac{1}{2^{j+1}} & \text{for } 2^j + 1 \leq r \leq 2^{j+1} - 1. \end{cases}$$

Consequently, the matrix G_{j+1} can be computed to be

$$G_{j+1} = \frac{1}{2^{j+1}} I_{2^{j+1}} - \frac{1}{2^{2j+3}} \chi_{j+1},$$

where $I_{2^{j+1}}$ is an identity matrix and $\chi_{j+1} = ((-1)^{r+s})_{r,s}$ is a checkerboard matrix, both of dimension 2^{j+1} . The eigenvalues of G_{j+1}^{-1} are just the reciprocals $\gamma_{j+1,r}^{-1}$ and we obtain

$$G_{j+1}^{-1} = 2^{j+1} I_{2^{j+1}} + \chi_{j+1}.$$

As bases of the orthogonal complementary spaces W_j , the wavelet functions on different levels are orthogonal, but the different translates on one level are not. Following Chui [1], we will therefore call them semi-orthogonal.

THEOREM 2.3 [22]. *For any $j \in \mathbb{N}_0$, the inner product matrix for the wavelets $H_{j+1} = (\langle \psi_{j,r}, \psi_{j,s} \rangle)_{r,s=0,\dots,2^{j+1}-1}$ can be factorized as*

$$H_{j+1} = \bar{F}_{j+1} \eta_{j+1} F_{j+1},$$

with $\eta_{j+1} = \text{diag}(\eta_{j+1,r})_{r=0,\dots,2^{j+1}-1}$, where the eigenvalues are

$$\eta_{j+1,r} = \begin{cases} \frac{1}{2^{j+2}} & \text{for } r = 0, \\ \frac{1}{2^{j+1}} & \text{for } 1 \leq r \leq 2^j - 1, \\ \frac{1}{2^{j+2}} & \text{for } r = 2^j, \\ \frac{1}{2^{j+1}} & \text{for } 2^j + 1 \leq r \leq 2^{j+1} - 1. \end{cases}$$

Thus, the matrix H_{j+1} can be written as

$$H_{j+1} = \frac{1}{2^{j+1}} I_{2^{j+1}} - \frac{1}{2^{2j+3}} 1_{j+1} - \frac{1}{2^{2j+3}} \chi_{j+1},$$

where $I_{2^{j+1}}$ is an identity matrix, 1_{j+1} is a matrix with constant entry one, and $\chi_{j+1} = ((-1)^{r+s})_{r,s}$ is a checkerboard matrix, all of dimension 2^{j+1} . One also obtains H_{j+1}^{-1} as

$$H_{j+1}^{-1} = 2^{j+1} I_{2^{j+1}} + 1_{j+1} + \chi_{j+1}.$$

Let $\underline{\phi}_{j+1}$ denote the vector $(\phi_{j,0}, \phi_{j,1}, \dots, \phi_{j,2^{j+1}-1})^T$ and, analogously, $\underline{\psi}_{j+1}$ the vector $(\psi_{j,0}, \psi_{j,1}, \dots, \psi_{j,2^{j+1}-1})^T$. Furthermore, we define a reordering for the vector of scaling functions by $\Pi_{j+2} \underline{\phi}_{j+2} = (\phi_{j+1,0}, \phi_{j+1,2}, \dots, \phi_{j+1,2m}, \dots, \phi_{j+1,2^{j+2}-2}, \phi_{j+1,1}, \phi_{j+1,3}, \dots, \phi_{j+1,2m+1}, \dots, \phi_{j+1,2^{j+2}-1})^T$, i.e., Π_{j+2} is chosen to be the suitable permutation matrix for this ordering.

Following [19], the two-scale relations can be expressed as

$$\begin{pmatrix} \underline{\phi}_{j+1} \\ \underline{\psi}_{j+1} \end{pmatrix} = R_{j+2} \Pi_{j+2} \underline{\phi}_{j+2}, \quad (2.6)$$

where the reconstruction matrix R_{j+2} has the form

$$R_{j+2} = \begin{pmatrix} I_{2^{j+1}} & K_{j+1} \\ -K_{j+1}^T & I_{2^{j+1}} \end{pmatrix}, \quad (2.7)$$

where K_{j+1} is a knot evaluation matrix consisting of the values of the scaling functions $\phi_{j,r}$ at the midpoints of the interpolation nodes $s\pi/2^j$, i.e.,

$$K_{j+1} = \left(\phi_{j,r} \left(\frac{(2s+1)\pi}{2^{j+1}} \right) \right)_{r,s=0}^{2^{j+1}-1}. \quad (2.8)$$

Consequently, R_{j+2} is a matrix of dimension 2^{j+2} with four circulant blocks.

On the other hand, the decomposition relations can be expressed as

$$\Pi_{j+2} \underline{\phi}_{j+2} = D_{j+2} \begin{pmatrix} \underline{\phi}_{j+1} \\ \underline{\psi}_{j+1} \end{pmatrix}, \quad (2.9)$$

using the decomposition matrix

$$\begin{aligned}
D_{j+2} &= R_{j+2}^{-1} = \begin{pmatrix} \tilde{G}_{j+1}^{-1} & -\tilde{G}_{j+1}^{-1}K_{j+1} \\ K_{j+1}^T \tilde{G}_{j+1}^{-1} & \tilde{G}_{j+1}^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \tilde{G}_{j+1}^{-1} & 0 \\ 0 & \tilde{G}_{j+1}^{-1} \end{pmatrix} \begin{pmatrix} I_{2^{j+1}} & -K_{j+1} \\ K_{j+1}^T & I_{2^{j+1}} \end{pmatrix}, \quad (2.10)
\end{aligned}$$

containing a slight modification of the inverse of the inner product matrix G_{j+1} , namely

$$\tilde{G}_{j+1}^{-1} = \frac{1}{2^{j+2}} G_{j+1}^{-1}.$$

For future implementation purposes, we conclude this section by stating the factorizations of the relevant submatrices of R_{j+2} and D_{j+2} , i.e., basically their eigenvalues.

Clearly, \tilde{G}_{j+1}^{-1} can be written as

$$\tilde{G}_{j+1}^{-1} = \bar{F}_{j+1} \tilde{\Gamma}_{j+1}^{-1} F_{j+1},$$

with

$$\tilde{\Gamma}_{j+1}^{-1} = \frac{1}{2} \text{diag}(1, \dots, 1, \underbrace{2}_{2^j}, 1, \dots, 1). \quad (2.11)$$

Furthermore, for the knot evaluation matrix K_{j+1} , one obtains (see [22]) $K_{j+1} = \bar{F}_{j+1} \alpha_{j+1} F_{j+1}$ and $K_{j+1}^T = \bar{F}_{j+1} \bar{\alpha}_{j+1} F_{j+1}$, with

$$\alpha_{j+1,s} = \begin{cases} \exp\left(\frac{s\pi}{2^{j+1}}i\right) & \text{for } s = 0, \dots, 2^j - 1, \\ 0 & \text{for } s = 2^j, \\ -\exp\left(\frac{s\pi}{2^{j+1}}i\right) & \text{for } s = 2^j + 1, \dots, 2^{j+1} - 1. \end{cases} \quad (2.12)$$

The following matrix/vector equations involving K_{j+1} and K_{j+1}^T can be obtained by interpreting their factorizations as eigenvector/eigenvalue relations. For $s = 0, \dots, 2^j - 1$, it holds that

$$K_{j+1} \left(\cos \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = \left(\cos \frac{(2r-1)s\pi}{2^{j+1}} \right)_{r=0}^{2^{j+1}-1} \quad (2.13)$$

and

$$K_{j+1} \left(\sin \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = \left(\sin \frac{(2r-1)s\pi}{2^{j+1}} \right)_{r=0}^{2^{j+1}-1},$$

as well as

$$K_{j+1}^T \left(\cos \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = \left(\cos \frac{(2r+1)s\pi}{2^{j+1}} \right)_{r=0}^{2^{j+1}-1}$$

and

$$K_{j+1}^T \left(\sin \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = \left(\sin \frac{(2r+1)s\pi}{2^{j+1}} \right)_{r=0}^{2^{j+1}-1} \quad (2.14)$$

For $s = 2^j$, we have

$$K_{j+1} \left(\cos \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = K_{j+1}^T \left(\cos \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = 0. \quad (2.15)$$

Finally, for $s = 2^j + 1, \dots, 2^{j+1} - 1$,

$$K_{j+1} \left(\cos \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = - \left(\cos \frac{(2r-1)s\pi}{2^{j+1}} \right)_{r=0}^{2^{j+1}-1} \quad (2.16)$$

and

$$K_{j+1} \left(\sin \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = - \left(\sin \frac{(2r-1)s\pi}{2^{j+1}} \right)_{r=0}^{2^{j+1}-1},$$

as well as

$$K_{j+1}^T \left(\cos \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = - \left(\cos \frac{(2r+1)s\pi}{2^{j+1}} \right)_{r=0}^{2^{j+1}-1} \quad (2.17)$$

and

$$K_{j+1}^T \left(\sin \frac{rs\pi}{2^j} \right)_{r=0}^{2^{j+1}-1} = - \left(\sin \frac{(2r+1)s\pi}{2^{j+1}} \right)_{r=0}^{2^{j+1}-1}.$$

3. MATRIX REPRESENTATION OF WAVELET DECOMPOSITION AND RECONSTRUCTION

For a given $N \in \mathbb{N}_0$, as $V_{N+1} = V_N \oplus W_N$, a function $f_{N+1} \in V_{N+1}$ can be written uniquely as

$$f_{N+1}(x) = f_N(x) + g_N(x), \quad \text{with } f_N \in V_N \text{ and } g_N \in W_N.$$

Using the basis functions of these spaces, one obtains

$$f_{N+1}(x) = \sum_{s=0}^{2^{N+2}-1} c_{N+2,s} \phi_{N+1,s}(x),$$

$$f_N(x) = \sum_{s=0}^{2^{N+1}-1} c_{N+1,s} \phi_{N,s}(x) \quad \text{and}$$

$$g_N(x) = \sum_{s=0}^{2^{N+1}-1} d_{N+1,s} \psi_{N,s}(x).$$

Denoting the respective coefficient vectors by $\underline{c}_{N+1}^T = (c_{N+1,0}, c_{N+1,1}, \dots, c_{N+1,2^{N+1}-1})$ and $\underline{d}_{N+1}^T = (d_{N+1,0}, d_{N+1,1}, \dots, d_{N+1,2^{N+1}-1})$, and using the basis vectors defined for (2.6), this means

$$\underline{c}_{N+2}^T \underline{\phi}_{N+2} = \underline{c}_{N+1}^T \underline{\phi}_{N+1} + \underline{d}_{N+1}^T \underline{\psi}_{N+1}.$$

As $\underline{c}_{N+2}^T \underline{\phi}_{N+2} = (\Pi_{N+2} \underline{c}_{N+2})^T \Pi_{N+2} \underline{\phi}_{N+2}$, the matrix form (2.9) of the decomposition relation yields

$$\begin{aligned}
f_{N+1} &= (\Pi_{N+2}\underline{c}_{N+2})^T \Pi_{N+2}\underline{\phi}_{N+2} \\
&= (\Pi_{N+2}\underline{c}_{N+2})^T D_{N+2} \begin{pmatrix} \underline{\phi}_{N+1} \\ \underline{\psi}_{N+1} \end{pmatrix}.
\end{aligned}$$

On the other hand, it holds that

$$f_N + g_N = (\underline{c}_{N+1}^T \underline{d}_{N+1}^T) \begin{pmatrix} \underline{\phi}_{N+1} \\ \underline{\psi}_{N+1} \end{pmatrix}.$$

Comparing coefficients leads to

$$(\underline{c}_{N+1}^T \underline{d}_{N+1}^T) = (\Pi_{N+2}\underline{c}_{N+2})^T D_{N+2},$$

and taking the transpose finally gives the matrix form of one step of the *decomposition algorithm*

$$\begin{pmatrix} \underline{c}_{N+1} \\ \underline{d}_{N+1} \end{pmatrix} = D_{N+2}^T \Pi_{N+2} \underline{c}_{N+2}. \quad (3.1)$$

Multiplying by the inverse $(D_{N+2}^T)^{-1} = R_{N+2}^T$ yields the matrix representation of one step of the *reconstruction algorithm*

$$\underline{c}_{N+2} = \Pi_{N+2}^T R_{N+2}^T \begin{pmatrix} \underline{c}_{N+1} \\ \underline{d}_{N+1} \end{pmatrix}. \quad (3.2)$$

Using the eigenvalues from (2.11) and (2.12), the reconstruction and decomposition steps can be factorized as

$$\begin{aligned}
\Pi_{N+2}\underline{c}_{N+2} &= \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \begin{pmatrix} I_{2^{N+1}} & -\alpha_{N+1} \\ \bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \begin{pmatrix} \underline{c}_{N+1} \\ \underline{d}_{N+1} \end{pmatrix} \quad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} \underline{c}_{N+1} \\ \underline{d}_{N+1} \end{pmatrix} &= \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_{N+1}^{-1} & 0 \\ 0 & \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} I_{2^{N+1}} & \alpha_{N+1} \\ -\bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \\
&\quad \times \Pi_{N+2}\underline{c}_{N+2}. \quad (3.4)
\end{aligned}$$

Consequently, two consecutive reconstruction steps can be written as

$$\underline{c}_{N+2} = \Pi_{N+2}^T R_{N+2}^T \begin{pmatrix} \Pi_{N+1}^T R_{N+1}^T & 0 \\ 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} \underline{c}_N \\ \underline{d}_N \end{pmatrix} \quad (3.5)$$

and two decomposition steps as

$$\begin{pmatrix} \underline{c}_N \\ \underline{d}_N \end{pmatrix} = \begin{pmatrix} D_{N+1}^T \Pi_{N+1} & 0 \\ 0 & I_{2^{N+1}} \end{pmatrix} D_{N+2}^T \Pi_{N+2} \underline{c}_{N+2}. \quad (3.6)$$

Using the factorization (3.3), the two reconstruction steps in (3.5) turn into

$$\begin{aligned}
\underline{c}_{N+2} &= \Pi_{N+2}^T \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \begin{pmatrix} I_{2^{N+1}} & -\alpha_{N+1} \\ \bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \begin{pmatrix} \Pi_{N+1}^T & 0 \\ 0 & I_{2^{N+1}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \bar{F}_N & 0 & 0 \\ 0 & \bar{F}_N & 0 \\ 0 & 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} I_{2^N} & -\alpha_N & 0 \\ \bar{\alpha}_N & I_{2^N} & 0 \\ 0 & 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} F_N & 0 & 0 \\ 0 & F_N & 0 \\ 0 & 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} \underline{c}_N \\ \underline{d}_N \end{pmatrix}
\end{aligned}$$

and, with (3.4), the two decomposition steps (3.6) become

$$\begin{aligned}
\begin{pmatrix} \underline{c}_N \\ \underline{d}_N \end{pmatrix} &= \begin{pmatrix} \bar{F}_N & 0 & 0 \\ 0 & \bar{F}_N & 0 \\ 0 & 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_N^{-1} & 0 & 0 \\ 0 & \tilde{\Gamma}_N^{-1} & 0 \\ 0 & 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} I_{2^N} & \alpha_N & 0 \\ -\bar{\alpha}_N & I_{2^N} & 0 \\ 0 & 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} F_N & 0 & 0 \\ 0 & F_N & 0 \\ 0 & 0 & I_{2^{N+1}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \Pi_{N+1} & 0 \\ 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_{N+1}^{-1} & 0 \\ 0 & \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix} \begin{pmatrix} I_{2^{N+1}} & \alpha_{N+1} \\ -\bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \Pi_{N+2}\underline{c}_{N+2}.
\end{aligned}$$

In view of an implementation of decomposition and reconstruction using fast Fourier transform (FFT) techniques, the number of operations can be reduced by precomputing some of the above matrix products and storing their results for further use. These precomputed products are represented by the *transition matrices* for the decomposition algorithm

$$T_{N+1} := \begin{pmatrix} F_N & 0 \\ 0 & F_N \end{pmatrix} \Pi_{N+1} \bar{F}_{N+1},$$

$$\hat{T}_{N+2} := \begin{pmatrix} \tilde{\Gamma}_{N+1}^{-1} & \tilde{\Gamma}_{N+1}^{-1} \alpha_{N+1} \\ -\tilde{\Gamma}_{N+1}^{-1} \bar{\alpha}_{N+1} & \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix}$$

and

$$\tilde{T}_{N+2} := \begin{pmatrix} T_{N+1} & 0 \\ 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_{N+1}^{-1} & 0 \\ 0 & \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix} \begin{pmatrix} I_{2^{N+1}} & \alpha_{N+1} \\ -\bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix}$$

$$= \begin{pmatrix} T_{N+1} \tilde{\Gamma}_{N+1}^{-1} & T_{N+1} \tilde{\Gamma}_{N+1}^{-1} \alpha_{N+1} \\ -\tilde{\Gamma}_{N+1}^{-1} \bar{\alpha}_{N+1} & \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix},$$

and the *transition matrices* for the reconstruction algorithm

$$U_{N+1} := T_{N+1}^{-1} = F_{N+1} \Pi_{N+1}^T \begin{pmatrix} \bar{F}_N & 0 \\ 0 & \bar{F}_N \end{pmatrix}$$

and

$$\tilde{U}_{N+2} := \begin{pmatrix} I_{2^{N+1}} & -\alpha_{N+1} \\ \bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} U_{N+1} & 0 \\ 0 & I_{2^{N+1}} \end{pmatrix}$$

$$= \begin{pmatrix} U_{N+1} & -\alpha_{N+1} \\ \bar{\alpha}_{N+1} U_{N+1} & I_{2^{N+1}} \end{pmatrix}.$$

Explicit expressions for the transition matrices T_{N+1} , \hat{T}_{N+2} , \tilde{T}_{N+2} , U_{N+1} , and \tilde{U}_{N+2} are given in

LEMMA 3.1. *For any $N \in \mathbb{N}_0$, the matrices T_{N+1} and U_{N+1} can be computed to be*

$$T_{N+1} = \frac{\sqrt{2}}{2} \begin{pmatrix} I_{2^N} & I_{2^N} \\ \bar{E}_N^{(2)} & -\bar{E}_N^{(2)} \end{pmatrix} \quad \text{and}$$

$$U_{N+1} = \frac{\sqrt{2}}{2} \begin{pmatrix} I_{2^N} & E_N^{(2)} \\ I_{2^N} & -E_N^{(2)} \end{pmatrix}.$$

Also, the matrices \hat{T}_{N+2} , \tilde{T}_{N+2} , and \tilde{U}_{N+2} are given as

$$\hat{T}_{N+2} = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} I_{2^N} & 0 \\ 0 & I_{2^N}^* \end{pmatrix} & \begin{pmatrix} E_N^{(1)} & 0 \\ 0 & -iE_N^{(1)*} \end{pmatrix} \\ \begin{pmatrix} -\bar{E}_N^{(1)} & 0 \\ 0 & -i\bar{E}_N^{(1)*} \end{pmatrix} & \begin{pmatrix} I_{2^N} & 0 \\ 0 & I_{2^N}^* \end{pmatrix} \end{pmatrix},$$

$$\tilde{T}_{N+2} = \begin{pmatrix} \frac{\sqrt{2}}{4} \begin{pmatrix} I_{2^N} & I_{2^N}^* \\ \bar{E}_N^{(2)} & -\bar{E}_N^{(2)*} \end{pmatrix} & \frac{\sqrt{2}}{4} \begin{pmatrix} E_N^{(1)} & -iE_N^{(1)*} \\ \bar{E}_N^{(1)} & i\bar{E}_N^{(1)*} \end{pmatrix} \\ \frac{1}{2} \begin{pmatrix} -\bar{E}_N^{(1)} & 0 \\ 0 & -i\bar{E}_N^{(1)*} \end{pmatrix} & \frac{1}{2} \begin{pmatrix} I_{2^N} & 0 \\ 0 & I_{2^N}^* \end{pmatrix} \end{pmatrix},$$

and

$$\tilde{U}_{N+2} = \begin{pmatrix} \frac{\sqrt{2}}{2} \begin{pmatrix} I_{2^N} & E_N^{(2)} \\ I_{2^N} & -E_N^{(2)} \end{pmatrix} & \begin{pmatrix} -E_N^{(1)} & 0 \\ 0 & iE_N^{(1)*} \end{pmatrix} \\ \frac{\sqrt{2}}{2} \begin{pmatrix} \bar{E}_N^{(1)} & E_N^{(1)} \\ i\bar{E}_N^{(1)*} & -iE_N^{(1)*} \end{pmatrix} & I_{2^{N+1}} \end{pmatrix},$$

where

$$E_N^{(\ell)} := \text{diag} \left(\exp \left(\frac{\ell r \pi i}{2^{N+1}} \right) \right)_{r=0, \dots, 2^N-1} \quad (\ell = 1, 2, 3),$$

$$E_N^{(\ell)*} := \text{diag} \left(0, \exp \left(\frac{\ell r \pi i}{2^{N+1}} \right) \right)_{r=1, \dots, 2^N-1} \quad (\ell = 1, 3),$$

$$E_N^{(2)*} := \text{diag} \left(2, \exp \left(\frac{r \pi i}{2^N} \right) \right)_{r=1, \dots, 2^N-1},$$

and I_N^* is a diagonal matrix of dimension 2^N with $I_N^* := \text{diag}(2, 1, \dots, 1)$.

Proof. A direct computation using the definition of the Fourier matrix shows that

$$\Pi_{N+1} \bar{F}_{N+1} = \frac{\sqrt{2}}{2} \begin{pmatrix} \bar{F}_N & \bar{F}_N \\ \bar{F}_N \bar{E}_N^{(2)} & -\bar{F}_N \bar{E}_N^{(2)} \end{pmatrix},$$

giving the desired structure of T_{N+1} . One checks directly that the matrix given for U_{N+1} in the statement of the lemma is indeed the inverse of T_{N+1} . The formula for \hat{T}_{N+2} is given by (2.11) and (2.12), while the ones for \tilde{T}_{N+2} and \tilde{U}_{N+2} then result from straightforward matrix multiplications and some simplifications of the exponential terms. ■

The factorized version of ℓ reconstruction steps ($\ell \geq 2$) is

$$\begin{aligned} \underline{c}_{N+2} &= \Pi_{N+2}^T \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \tilde{U}_{N+2} \cdots \begin{pmatrix} \tilde{U}_{N-\ell+5} & 0 \\ 0 & I_{2^{N+2}-2^{N-\ell+5}} \end{pmatrix} \begin{pmatrix} \tilde{U}_{N-\ell+4} & 0 \\ 0 & I_{2^{N+2}-2^{N-\ell+4}} \end{pmatrix} \\ &\quad \times \begin{pmatrix} I_{2^{N-\ell+2}} & -\alpha_{N-\ell+2} & 0 \\ -\tilde{\alpha}_{N-\ell+2} & I_{2^{N-\ell+2}} & 0 \\ 0 & 0 & I_{2^{N+2}-2^{N-\ell+3}} \end{pmatrix} \begin{pmatrix} F_{N-\ell+2} & 0 & 0 \\ 0 & F_{N-\ell+2} & 0 \\ 0 & 0 & F_{N-\ell+3} \end{pmatrix} \begin{pmatrix} \underline{c}_{N-\ell+2} \\ \underline{d}_{N-\ell+2} \\ \underline{d}_{N-\ell+3} \\ \vdots \\ \underline{d}_{N+1} \end{pmatrix}, \end{aligned}$$

and using

$$\begin{pmatrix} F_N & 0 & 0 \\ 0 & F_N & 0 \\ 0 & 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} \Pi_{N+1} & 0 \\ 0 & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_{N+1}^{-1} & 0 \\ 0 & \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix} \begin{pmatrix} I_{2^{N+1}} & \alpha_{N+1} \\ -\tilde{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} = \begin{pmatrix} I_{2^{N+1}} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \tilde{T}_{N+2},$$

the one for ℓ decomposition steps is

$$\begin{aligned} \begin{pmatrix} \underline{c}_{N-\ell+2} \\ \underline{d}_{N-\ell+2} \\ \underline{d}_{N-\ell+3} \\ \vdots \\ \underline{d}_{N+1} \end{pmatrix} &= \begin{pmatrix} \bar{F}_{N-\ell+2} & 0 & 0 \\ 0 & \bar{F}_{N-\ell+2} & 0 \\ 0 & 0 & \bar{F}_{N-\ell+3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_{N-\ell+2}^{-1} & \tilde{\Gamma}_{N-\ell+2}^{-1} \alpha_{N-\ell+2} & 0 \\ -\tilde{\Gamma}_{N-\ell+2}^{-1} \alpha_{N-\ell+2} & \tilde{\Gamma}_{N-\ell+2}^{-1} & 0 \\ 0 & 0 & I_{2^{N+2}-2^{N-\ell+3}} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \tilde{T}_{N-\ell+4} & 0 \\ 0 & I_{2^{N+2}-2^{N-\ell+4}} \end{pmatrix} \begin{pmatrix} \tilde{T}_{N-\ell+5} & 0 \\ 0 & I_{2^{N+2}-2^{N-\ell+5}} \end{pmatrix} \cdots \tilde{T}_{N+2} \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \Pi_{N+2} \underline{c}_{N+2}. \end{aligned}$$

Therefore, we have

ALGORITHM A1. Decomposition using semiorthogonal trigonometric wavelets.

Input:

$N \in \mathbb{N}$ Starting level (sufficiently large).

\underline{c}_{N+2}^0 Coefficient vector of the initial approximation $f_{N+1} \in V_{N+1}$. These coefficients are the 2^{N+2} function values taken at equidistant knots in the interval $[0, 2\pi)$.

ℓ Number of decomposition levels ($1 \leq \ell \leq N$).

(i) Reorder the elements of the vector \underline{c}_{N+2}^0 into a vector

$$\begin{pmatrix} \underline{c}_{N+1}^e \\ \underline{c}_{N+1}^o \end{pmatrix} := \Pi_{N+2} \underline{c}_{N+2}^0,$$

whereby $\underline{c}_{N+1}^e(\underline{c}_{N+1}^o)$ contains the elements of \underline{c}_{N+2}^0 with even (odd) indices.

(ii) Compute two Fourier transformations (of length 2^{N+1}) of the vectors \underline{c}_{N+1}^e and \underline{c}_{N+1}^o and set

$$\underline{c}_{N+2} := \begin{pmatrix} F_{N+1} \underline{c}_{N+1}^e \\ F_{N+1} \underline{c}_{N+1}^o \end{pmatrix}.$$

(iii) For each level $j(j = 0, \dots, \ell - 2)$ do

• compute $\tilde{\underline{c}}_{N+2-j} := \tilde{T}_{N+2-j} \underline{c}_{N+2-j}$ and split

$$\tilde{\underline{c}}_{N+2-j} =: \begin{pmatrix} \underline{c}_{N+1-j} \\ \tilde{\underline{d}}_{N+1-j} \end{pmatrix}$$

• compute one inverse Fourier transformation (of length 2^{N+1-j})

$$\underline{d}_{N+1-j} := \bar{F}_{N+1-j} \tilde{\underline{d}}_{N+1-j}.$$

(iv) Compute $\tilde{\underline{c}}_{N-\ell+3} := \hat{T}_{N-\ell+3} \underline{c}_{N-\ell+3}$ and split

$$\tilde{\underline{c}}_{N-\ell+3} =: \begin{pmatrix} \tilde{\underline{c}}_{N-\ell+2} \\ \tilde{\underline{d}}_{N-\ell+2} \end{pmatrix}.$$

(v) Compute two inverse Fourier transformations (of length $2^{N-\ell+2}$)

$$\underline{c}_{N+2-\ell} := \bar{F}_{N+2-\ell} \tilde{\underline{c}}_{N+2-\ell} \quad \text{and} \quad \underline{d}_{N+2-\ell} := \bar{F}_{N+2-\ell} \tilde{\underline{d}}_{N+2-\ell}$$

Output: Vector of the wavelet coefficients and lowest level scaling function coefficients $(\underline{c}_{N-\ell+2}, \underline{d}_{N-\ell+2}, \underline{d}_{N-\ell+3}, \dots, \underline{d}_{N+1})^T$ and the corresponding

ALGORITHM A2. Reconstruction using semiorthogonal trigonometric wavelets.

Input:

$N \in \mathbb{N}$ Starting level (determined by the decomposition algorithm A1).

$(\underline{c}_{N-\ell+2}, \underline{d}_{N-\ell+2}, \underline{d}_{N-\ell+3}, \dots, \underline{d}_{N+1})^T$ Coefficients from the decomposition algorithm A1.

ℓ Number of reconstruction levels ($1 \leq \ell \leq N$).

(i) Compute two Fourier transformations (of length $2^{N-\ell+2}$) of the vectors $\underline{c}_{N-\ell+2}$ and $\underline{d}_{N-\ell+2}$, i.e., $\hat{\underline{c}}_{N-\ell+2} := F_{N-\ell+2} \underline{c}_{N-\ell+2}$, $\hat{\underline{d}}_{N-\ell+2} := F_{N-\ell+2} \underline{d}_{N-\ell+2}$.

(ii) Compute

$$\tilde{\underline{c}}_{N-\ell+3} := \begin{pmatrix} \hat{\underline{c}}_{N-\ell+2} - \alpha_{N-\ell+2} \hat{\underline{d}}_{N-\ell+2} \\ -\bar{\alpha}_{N-\ell+2} \hat{\underline{c}}_{N-\ell+2} + \hat{\underline{d}}_{N-\ell+2} \end{pmatrix}.$$

(iii) For each level $j(j = \ell - 2, \dots, 0)$ do

- compute one Fourier transformation (of length 2^{N+1-j})

- compute

$$\tilde{\underline{c}}_{N+2-j} := \tilde{U}_{N+2-j} \begin{pmatrix} \tilde{\underline{c}}_{N+1-j} \\ \hat{\underline{d}}_{N+1-j} \end{pmatrix}.$$

(iv) Compute two inverse Fourier transformations (of length 2^{N+1}) of the first 2^{N+1} elements of $\tilde{\underline{c}}_{N+2}$ and of the last 2^{N+1} elements of $\tilde{\underline{c}}_{N+2}$, i.e.,

$$\underline{\tilde{c}}_{N+2}^0 := \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \tilde{\underline{c}}_{N+2}.$$

(v) Reorder the elements of the vector $\underline{\tilde{c}}_{N+2}^0$ into a vector \underline{c}_{N+2}^0 according to the rule

$$c_{N+2,i}^0 := \begin{cases} \tilde{c}_{N+2,i/2}^0 & \text{if } i \text{ is even,} \\ \tilde{c}_{N+2,2^{N+1}+(i-1)/2}^0 & \text{if } i \text{ is odd,} \end{cases}$$

$$(i = 0, \dots, 2^{N+2} - 1).$$

Output: Vector of the reconstructed coefficients \underline{c}_{N+2}^0 .

4. TRIGONOMETRIC WAVELET PACKET FUNCTIONS AND SPACES

In order to achieve a finer resolution of sine and cosine frequencies in the wavelet spaces W_j , we now proceed to define trigonometric wavelet packet functions following the general ideas as introduced and studied by Coifman, Meyer,

Quake, and Wickerhauser (see [29] and the list of references therein), appropriately modified for a trigonometric MRA.

For a chosen $N \in \mathbb{N}_0$, let V_N be an initial (high level) sample space of dimension 2^{N+1} , i.e.,

$$V_N = \text{span}\{1, \cos x, \dots, \cos(2^N - 1)x, \cos 2^N x,$$

$$\sin x, \dots, \sin(2^N - 1)x\}.$$

The goal is to break up the space V_N into 2^ℓ wavelet packet spaces of dimension $2^{N+1-\ell}$, with $\ell = 1, \dots, N$. We start by describing the packet functions spanning the packet spaces.

DEFINITION 4.1. For given $N \in \mathbb{N}_0$ and order $\ell = 1, \dots, N$, consider for $k = 0, \dots, 2^\ell - 1$ the dyadic expansion of k , i.e.,

$$k = \sum_{r=1}^{\ell} \epsilon_{r-1}^{(k)} \cdot 2^{r-1}, \quad \epsilon_{r-1}^{(k)} = \begin{cases} 0 \\ 1 \end{cases}.$$

The vector of the k th wavelet packet functions of order ℓ and initial level N is defined as

$$\begin{pmatrix} \psi_{N-\ell,0}^{(k)} \\ \vdots \\ \psi_{N-\ell,2^{N+1-\ell}-1}^{(k)} \end{pmatrix}$$

$$:= Q_{N-\ell+2,\epsilon_0^{(k)}} \cdots Q_{N,\epsilon_{\ell-2}^{(k)}} Q_{N+1,\epsilon_{\ell-1}^{(k)}} \begin{pmatrix} \phi_{N,0} \\ \vdots \\ \phi_{N,2^{N+1}-1} \end{pmatrix},$$

with

$$Q_{s+1,0} = (I_{2^s} \ K_s) \Pi_{s+1} \quad \text{and} \quad Q_{s+1,1} = (-K_s^T \ I_{2^s}) \Pi_{s+1}$$

for $s = N, \dots, N - \ell + 1$.

This actually means that the k th wavelet packet function is defined by starting from the N th level scaling functions and iteratively applying either the scaling function or wavelet two-scale coefficients, depending on whether the corresponding coefficient in the dyadic expansion of k is zero or one. Note that this implies

$$\psi_{N-\ell,n}^{(k)}(x) = \psi_{N-\ell,0}^{(k)}\left(x - n \frac{\pi}{2^{N-\ell}}\right).$$

Alternatively, Definition 4.1 can be rephrased in terms of a recursion by writing the wavelet packet functions of order ℓ as linear combinations of the wavelet packet functions of order $\ell - 1$ using the corresponding reconstruction

matrix, i.e., for any $k = 0, \dots, 2^{\ell-1} - 1$,

$$\begin{pmatrix} \psi_{N-\ell,0}^{(2k)} \\ \vdots \\ \psi_{N-\ell,2^{N+1-\ell}-1}^{(2k)} \\ \psi_{N-\ell,0}^{(2k+1)} \\ \vdots \\ \psi_{N-\ell,2^{N+1-\ell}-1}^{(2k+1)} \end{pmatrix} = R_{N+2-\ell} \Pi_{N+2-\ell} \begin{pmatrix} \psi_{N-\ell+1,0}^{(k)} \\ \vdots \\ \psi_{N-\ell+1,2^{N+2-\ell}-1}^{(k)} \end{pmatrix}, \quad (4.1)$$

where the original scaling functions and wavelets $\phi_{j,k}$ and $\psi_{j,k}$ are renamed $\psi_{j,k}^{(0)}$ and $\psi_{j,k}^{(1)}$ for uniformity. Note that for $k = 0$, we recover the original two-scale relations (2.6).

Next, it is established that all wavelet packet functions possess interpolatory properties which are related to the ones of the original scaling functions (see (2.3)) and wavelets (see (2.5)).

LEMMA 4.1. *For given $N \in \mathbb{N}_0$, order $\ell = 1, \dots, N$, and packet number $k = 0, \dots, 2^\ell - 1$, the functions $\psi_{N-\ell,n}^{(k)}$ possess the following interpolatory property for all indices $n, r = 0, \dots, 2^{N+1-\ell} - 1$:*

$$\psi_{N-\ell,n}^{(k)} \left(\frac{r\pi}{2^{N-\ell}} + \sum_{s=1}^{\ell} \epsilon_{s-1}^{(k)} \frac{\pi}{2^{N-\ell+s}} \right) = \delta_{n,r}.$$

Proof. This lemma is proved by induction on the order ℓ of the packet functions, with (2.3) and (2.5) serving as initial steps. Assuming the result for packet functions of order $\ell - 1$, one obtains that for $i = 0, 1$ and $k = 0, \dots, 2^{\ell-1} - 1$

$$\begin{aligned} \frac{r\pi}{2^{N-\ell}} + \sum_{s=1}^{\ell} \epsilon_{s-1}^{(2k+i)} \frac{\pi}{2^{N-\ell+s}} &= \frac{2r}{2^{N-\ell+1}} + \sum_{s=0}^{\ell-1} \epsilon_s^{(2k+i)} \frac{\pi}{2^{N-\ell+1+s}} \\ &= \frac{2r+i}{2^{N-\ell+1}} + \sum_{s=0}^{\ell-1} \epsilon_{s-1}^{(k)} \frac{\pi}{2^{N-\ell+1+s}} \end{aligned}$$

for $n, r = 0, \dots, 2^{N+1-\ell} - 1$. Using this equality, one can now deduce the desired interpolatory property from the two-scale relations (4.1), invoking the induction hypothesis and the special structure of the matrix $R_{N+2-\ell}$ as given in (2.7). ■

We now introduce wavelet packet spaces as the spans of the respective packet functions.

DEFINITION 4.2. *For given $N \in \mathbb{N}_0$, order $\ell = 1, \dots, N$, and arbitrary packet number $k = 0, \dots, 2^\ell - 1$, define the k th wavelet packet space as*

$$W_{N-\ell}^{(k)} = \text{span}\{\psi_{N-\ell,n}^{(k)} : n = 0, \dots, 2^{N+1-\ell} - 1\}.$$

The interpolatory properties of Lemma 4.1 ensure that the functions $\psi_{N-\ell,n}^{(k)}$ form a basis of $W_{N-\ell}^{(k)}$. It remains to show that the packet spaces are mutually orthogonal.

THEOREM 4.1. *For given $N \in \mathbb{N}_0$ and order $\ell = 1, \dots, N$, the packet spaces constitute an orthogonal decomposition of the original sample space V_N :*

$$V_N = \bigoplus_{k=0}^{2^\ell-1} W_{N-\ell}^{(k)}.$$

The inner product matrices of the packet functions, i.e., $G_{N-\ell+1}^{(k)} := (\langle \psi_{N-\ell,r}^{(k)}, \psi_{N-\ell,s}^{(k)} \rangle)_{r,s}$, are given as

$$G_{N-\ell+1}^{(k)} = \begin{cases} G_{N-\ell+1}^{(1)}, & k = 1, \dots, 2^{\ell-1} - 1, \\ G_{N-\ell+1}^{(0)}, & k = 0, \end{cases}$$

with $G_{N-\ell+1}^{(0)} = G_{N-\ell+1}$ and $G_{N-\ell+1}^{(1)} = H_{N-\ell+1}$ given in Theorems 2.2 and 2.3.

Proof. The proof is again by induction on the order ℓ , with Theorem 2.1 furnishing the initial step. The recursion formula (4.1) together with the invertibility of the matrix $R_{N+2-\ell} \Pi_{N+2-\ell}$ establishes that for $k = 0, \dots, 2^{\ell-1} - 1$,

$$W_{N-\ell+1}^{(k)} = W_{N-\ell}^{(2k)} + W_{N-\ell}^{(2k+1)},$$

so that with the induction hypothesis, it is left to prove that for $r, s = 0, \dots, 2^{N-\ell+1} - 1$

$$\langle \psi_{N-\ell,r}^{(2k)}, \psi_{N-\ell,s}^{(2k+1)} \rangle = 0.$$

Using (4.1), we obtain

$$M := \begin{pmatrix} (\langle \psi_{N-\ell,r}^{(2k)}, \psi_{N-\ell,s}^{(2k)} \rangle)_{r,s} & (\langle \psi_{N-\ell,r}^{(2k)}, \psi_{N-\ell,s}^{(2k+1)} \rangle)_{r,s} \\ (\langle \psi_{N-\ell,r}^{(2k+1)}, \psi_{N-\ell,s}^{(2k)} \rangle)_{r,s} & (\langle \psi_{N-\ell,r}^{(2k+1)}, \psi_{N-\ell,s}^{(2k+1)} \rangle)_{r,s} \end{pmatrix}$$

$$= R_{N+2-\ell} \Pi_{N+2-\ell} G_{N+2-\ell}^{(k)} \Pi_{N+2-\ell}^T R_{N+2-\ell}^T.$$

The two-scale relations for the original scaling functions and wavelets imply that for $k = 0$,

$$R_{N+2-\ell} \Pi_{N+2-\ell} G_{N+2-\ell}^{(0)} \Pi_{N+2-\ell}^T R_{N+2-\ell}^T$$

$$= \begin{pmatrix} G_{N+1-\ell}^{(0)} & 0 \\ 0 & G_{N+1-\ell}^{(1)} \end{pmatrix}, \quad (4.2)$$

while for $k \neq 0$, by the induction hypothesis, Theorem 2.3, and (4.2),

$$\begin{aligned}
& R_{N+2-\ell} \Pi_{N+2-\ell} G_{N+2-\ell}^{(k)} \Pi_{N+2-\ell}^T R_{N+2-\ell}^T \\
&= R_{N+2-\ell} \Pi_{N+2-\ell} G_{N+2-\ell}^{(1)} \Pi_{N+2-\ell}^T R_{N+2-\ell}^T \\
&= R_{N+2-\ell} \Pi_{N+2-\ell} \left(G_{N+2-\ell}^{(0)} - \frac{1}{2^{2N+5-2\ell}} 1_{N+2-\ell} \right) \\
&\quad \times \Pi_{N+2-\ell}^T R_{N+2-\ell}^T \\
&= \begin{pmatrix} G_{N+1-\ell}^{(0)} & 0 \\ 0 & G_{N+1-\ell}^{(1)} \end{pmatrix} \\
&\quad - \frac{1}{2^{2N+5-2\ell}} R_{N+2-\ell} \Pi_{N+2-\ell} 1_{N+2-\ell} \Pi_{N+2-\ell}^T R_{N+2-\ell}^T.
\end{aligned}$$

The structure of $R_{N+2-\ell}$, as given in (2.7), combined with (2.13) and (2.14) for $s = 0$, i.e., $K_{N+1-\ell} 1_{N+1-\ell} = K_{N+1-\ell}^T 1_{N+1-\ell} = 1_{N+1-\ell}$, yields

$$M = \begin{pmatrix} G_{N+1-\ell}^{(0)} & 0 \\ 0 & G_{N+1-\ell}^{(1)} \end{pmatrix} - \frac{1}{2^{2N+3-2\ell}} \begin{pmatrix} 1_{N+1-\ell} & 0 \\ 0 & 0 \end{pmatrix}$$

and thus completes the proof. ■

In the following, the problems of how to determine which sine and cosine functions can be found in a given packet space, and conversely, for a given sine or cosine term, how to identify the packet spaces in which they can be found, are considered. It turns out that due to a swapping effect for high and low frequencies whenever the wavelet coefficients are applied in Definition 4.1, i.e., the matrices $Q_{j,1}$ are used, it is more complicated to keep track of the contents of the packets. Note that the ordering produced is not one of those listed in [29], such as Paley or sequency ordering.

For the vector $\underline{\epsilon}_\ell^{(k)} = (\epsilon_0^{(k)}, \dots, \epsilon_{\ell-1}^{(k)})^T$ given by the dyadic expansion of k in Definition 4.1, define another vector $\underline{\tilde{\epsilon}}_\ell^{(k)} = (\tilde{\epsilon}_0^{(k)}, \dots, \tilde{\epsilon}_{\ell-1}^{(k)})^T$ by recursively setting

$$\begin{aligned}
\tilde{\epsilon}_{\ell-1}^{(k)} &:= \epsilon_{\ell-1}^{(k)} \\
\tilde{\epsilon}_r^{(k)} &:= \begin{cases} \epsilon_r^{(k)} & \text{if } \epsilon_{r+1}^{(k)} = 0 \\ 1 - \epsilon_r^{(k)} & \text{if } \epsilon_{r+1}^{(k)} = 1 \end{cases} \quad \text{for } r = \ell - 2, \dots, 0. \quad (4.3)
\end{aligned}$$

Furthermore, if $k = \sum_{r=1}^{\ell} \epsilon_{r-1}^{(k)} \cdot 2^{r-1}$, then \bar{k} is defined as $\bar{k} := \sum_{r=1}^{\ell} \tilde{\epsilon}_{r-1}^{(k)} \cdot 2^{r-1}$. It is now possible to describe the set of sine/cosine terms in $W_{N-\ell}^{(k)}$. Note beforehand

that $W_{N-\ell}^{(0)} = V_{N-\ell} = \text{span}\{1, \cos x, \dots, \cos(2^{N-\ell} - 1)x, \cos 2^{N-\ell}x, \sin x, \dots, \sin(2^{N-\ell} - 1)x\}$.

THEOREM 4.2. *For given $N \in \mathbb{N}_0$, order $\ell = 1, \dots, N$, and packet number $k = 1, \dots, 2^\ell - 1$, it holds that*

$$W_{N-\ell}^{(k)} = \text{span} \left\{ \begin{array}{l} \sin(\bar{k}2^{N-\ell}x - u_k\pi), \\ \cos(\bar{k}2^{N-\ell}x + s)x, \sin(\bar{k}2^{N-\ell}x + s)x, \\ \text{for } s = 1, \dots, 2^{N-\ell} - 1, \\ \cos((\bar{k} + 1)2^{N-\ell}x - t_k\pi) \end{array} \right\},$$

where

$$\begin{aligned}
u_k &= \begin{cases} 0 & \text{if } \bar{k} \text{ is a power of 2} \\ (1 - \epsilon_0^{(k)} \cdot \tilde{\epsilon}_0^{(k)})2^{-1} + \bar{k} \left(\sum_{r=2}^{\ell} \epsilon_{r-1}^{(k)} \cdot 2^{-r} \right) & \text{otherwise} \end{cases} \\
t_k &= \begin{cases} 0 & \text{if } \bar{k} + 1 \text{ is a power of 2} \\ (\bar{k} + 1) \sum_{r=1}^{\ell} \epsilon_{r-1}^{(k)} \cdot 2^{-r} & \text{otherwise} \end{cases}
\end{aligned}$$

Proof. As usual, we proceed by induction on ℓ . For $\ell = 1$, there is only the case $k = 1$, with $W_{N-1}^{(1)} = \text{span}\{\cos(2^{N-1}+1)x, \dots, \cos(2^N-1)x, \cos 2^N x, \sin 2^{N-1}x, \dots, \sin(2^N-1)x\}$, which fits the statement of the theorem with $\bar{k} = 1, u_k = 0$, and $t_k = 0$.

Let the statement hold for $\ell - 1$. Then, $W_{N-\ell+1}^{(k)} = W_{N-\ell}^{(2k)} \oplus W_{N-\ell}^{(2k+1)}$, according to Theorem 4.1. Due to Lemma 4.1, the coefficient vector of a function $f \in W_{N-\ell+1}^{(k)}$ with respect to the packet basis $\{\psi_{N-\ell+1,r}^{(k)}\}$ is given by

$$\underline{f}_{N-\ell+2} = \left(f \left(\frac{r\pi}{2^{N-\ell+1}} + \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} \frac{\pi}{2^{N-\ell+1+s}} \right) \right)_{r=0}^{2^{N+2-\ell}-1}.$$

Consequently, a decomposition into its components from $W_{N-\ell}^{(2k)}$ and $W_{N-\ell}^{(2k+1)}$ amounts to a multiplication of the vector $\underline{f}_{N-\ell+2}$ by the matrix $D_{N+2-\ell}^T \Pi_{N+2-\ell}$:

$$\begin{aligned}
\begin{pmatrix} \underline{f}_{N+1-\ell}^{(2k)} \\ \underline{f}_{N+1-\ell}^{(2k+1)} \end{pmatrix} &= \begin{pmatrix} \tilde{G}_{N+1-\ell}^{-1} & 0 \\ 0 & \tilde{G}_{N+1-\ell}^{-1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} I_{2^{N+1-\ell}} & K_{N+1-\ell} \\ -K_{N+1-\ell}^T & I_{2^{N+1-\ell}} \end{pmatrix} \begin{pmatrix} \underline{f}_{N+1-\ell}^e \\ \underline{f}_{N+1-\ell}^o \end{pmatrix},
\end{aligned}$$

with

$$\underline{f}_{N-\ell+1}^e = \left(f \left(\frac{r\pi}{2^{N-\ell}} + \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} \frac{\pi}{2^{N-\ell+1+s}} \right) \right)_{r=0}^{2^{N+1-\ell}-1}$$

and

$$\begin{aligned}
&\underline{f}_{N-\ell+1}^o \\
&= \left(f \left(\frac{r\pi}{2^{N-\ell}} + \frac{\pi}{2^{N-\ell+1}} + \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} \frac{\pi}{2^{N-\ell+1+s}} \right) \right)_{r=0}^{2^{N+1-\ell}-1}.
\end{aligned}$$

The function f is in $W_{N-\ell}^{(2k+1)}$ if and only if $\underline{f}_{N+1-\ell}^{(2k)} = \underline{0}$. As the matrix $\tilde{G}_{N+1-\ell}^{-1}$ is non-singular, this amounts to

$$(I_{2^{N+1-\ell}} \ K_{N+1-\ell}) \begin{pmatrix} \underline{f}_{N+1-\ell}^e \\ \underline{f}_{N+1-\ell}^o \end{pmatrix} = \underline{0},$$

while f is in $W_{N-\ell}^{(2k)}$ if and only if $\underline{f}_{N+1-\ell}^{(2k+1)} = \underline{0}$, i.e.,

$$(-K_{N+1-\ell}^T \ I_{2^{N+1-\ell}}) \begin{pmatrix} \underline{f}_{N+1-\ell}^e \\ \underline{f}_{N+1-\ell}^o \end{pmatrix} = \underline{0}.$$

According to the induction hypothesis, the following functions span the space $W_{N-\ell+1}^{(k)}$: $\sin(\bar{k}2^{N-\ell+1}x - u_k\pi)$, $\cos(\bar{k}2^{N-\ell+1+s}x)$, $\sin(\bar{k}2^{N-\ell+1+s}x)$, for $s = 1, \dots, 2^{N-\ell+1} - 1$, and $\cos((\bar{k}+1)2^{N-\ell+1}x - t_k\pi)$. Now, it needs to be investigated whether they turn up in either $W_{N-\ell}^{(2k)}$ or $W_{N-\ell}^{(2k+1)}$ or are somehow split among the two. We have to distinguish two cases, namely \bar{k} is even and \bar{k} is odd. Note that for \bar{k} even, we have $2\bar{k} = 2\bar{k}$ and $2\bar{k} + 1 = 2\bar{k} + 1$, while for \bar{k} odd, $2\bar{k} = 2\bar{k} + 1$ and $2\bar{k} + 1 = 2\bar{k}$. In the following, we restrict ourselves to the case \bar{k} even; the other one necessitates similar computations with the same type of strategy.

A direct computation establishes the vectors $\underline{f}_{N-\ell+1}^e$ and $\underline{f}_{N-\ell+1}^o$ for the function $f(x) = \sin(\bar{k}2^{N-\ell+1}x - u_k\pi)$ as having all entries equal to $\sin(2\bar{k} \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} \frac{\pi}{2^{s+1}} - u_k\pi)$. Using the eigenvalue/eigenvector relation in (2.13) for $s = 0$ shows that $-K_{N+1-\ell}^T \underline{f}_{N-\ell+1}^e + \underline{f}_{N-\ell+1}^o = 0$, and therefore, $\sin(\bar{k}2^{N-\ell+1}x - u_k\pi) \in W_{N-\ell}^{(2k)}$. For \bar{k} even, $\bar{k}2^{N-\ell+1}x = 2\bar{k}2^{N-\ell}x$ and if \bar{k} is a power of two, implying $u_k = 0$, so is $2\bar{k}$ and $u_{2k} = 0$, too. Otherwise,

$$u_k = (1 - \epsilon_0^{(k)} \cdot \epsilon_0^{(k)})2^{-1} + \bar{k} \left(\sum_{s=2}^{\ell-1} \epsilon_{s-1}^{(k)} \cdot 2^{-s} \right) = u_{2k} + \frac{\bar{k}}{2} \epsilon_0^{(k)},$$

and, consequently, also $\sin(\bar{k}2^{N-\ell}x - u_{2k}\pi) \in W_{N-\ell}^{(2k)}$.

Now, for $f(x) = \cos(\bar{k}2^{N-\ell+1} + m)x$ with $m = 1, \dots, 2^{N+1-\ell} - 1$ (and similarly for the corresponding sine terms):

$$\underline{f}_{N-\ell+1}^e = \left(\cos c_m \cdot \cos \frac{rm\pi}{2^{N-\ell}} - \sin c_m \cdot \sin \frac{rm\pi}{2^{N-\ell}} \right)_{r=0}^{2^{N-\ell+1}}$$

and

$$\underline{f}_{N-\ell+1}^o = \left(\cos c_m \cdot \cos \frac{(2r+1)m\pi}{2^{N-\ell+1}} - \sin c_m \cdot \sin \frac{(2r+1)m\pi}{2^{N-\ell+1}} \right)_{r=0}^{2^{N-\ell+1}},$$

with $c_m = 2\bar{k} \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-s-1}\pi + m \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-N+\ell-s-1}\pi$, so that applying the relations in (2.14) shows that also

$\cos(\bar{k}2^{N-\ell} + m)x \in W_{N-\ell}^{(2k)}$ (and similarly $\sin(\bar{k}2^{N-\ell} + m)x \in W_{N-\ell}^{(2k)}$).

Finally, for $f(x) = \cos(\bar{k}2^{N-\ell+1} + 2^{N-\ell})x$ the corresponding vectors turn out to be

$$\underline{f}_{N-\ell+1}^e = \left((-1)^r \cos(2\bar{k} + 1) \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-s-1}\pi \right)_{r=0}^{2^{N-\ell+1}}$$

and

$$\underline{f}_{N-\ell+1}^o = \left(-(-1)^r \sin(2\bar{k} + 1) \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-s-1}\pi \right)_{r=0}^{2^{N-\ell+1}},$$

respectively, but these are eigenvectors of the matrices $K_{N+1-\ell}$ and $K_{N+1-\ell}^T$ for the eigenvalue zero (see (2.15)). Consequently, the function $\cos(2\bar{k} + 1)2^{N-\ell}x$ is neither in $W_{N-\ell}^{(2k)}$ nor in $W_{N-\ell}^{(2k+1)}$. The same holds for $g(x) = \sin(2\bar{k} + 1)2^{N-\ell}x$, for which

$$\underline{g}_{N-\ell+1}^e = \left((-1)^r \sin(2\bar{k} + 1) \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-s-1}\pi \right)_{r=0}^{2^{N-\ell+1}}$$

and

$$\underline{g}_{N-\ell+1}^o = \left((-1)^r \cos(2\bar{k} + 1) \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-s-1}\pi \right)_{r=0}^{2^{N-\ell+1}}.$$

Nevertheless,

$$\begin{aligned} & \left(\cos(2\bar{k} + 1) \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-s-1}\pi \right) \underline{f}_{N-\ell+1}^o \\ & + \left(\sin(2\bar{k} + 1) \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-s-1}\pi \right) \underline{g}_{N-\ell+1}^o = 0, \end{aligned}$$

meaning that the function

$$\begin{aligned} & \cos \left((\bar{k} + 1)2^{N-\ell}x - (\bar{k} + 1) \sum_{s=1}^{\ell-1} \epsilon_{s-1}^{(k)} 2^{-s-1}\pi \right) \\ & = \cos((\bar{k} + 1)2^{N-\ell}x - t_{2k}\pi) \in W_{N-\ell}^{(2k)}, \end{aligned}$$

thus establishing the spanning functions for $W_{N-\ell}^{(2k)}$, as desired.

With $\cos((\bar{k} + 1)2^{N-\ell}x - t_{2k}\pi) \in W_{N-\ell}^{(2k)}$, we also obtain $\sin((\bar{k} + 1)2^{N-\ell}x - t_{2k}\pi) \in W_{N-\ell}^{(2k+1)}$ and $t_{2k} = u_{2k+1}$, i.e., $\sin((\bar{k} + 1)2^{N-\ell}x - u_{2k+1}\pi) \in W_{N-\ell}^{(2k+1)}$. The functions $\cos((\bar{k} + 1)2^{N-\ell} + m)x$ and $\sin((\bar{k} + 1)2^{N-\ell} + m)x$ for $m = 1, \dots, 2^{N-\ell} - 1$ are orthogonal to the already established spanning functions of $W_{N-\ell}^{(2k)}$ and consequently are in $W_{N-\ell}^{(2k+1)}$. The same is true for $\cos((\bar{k} + 1)2^{N-\ell}x - t_k\pi)$. If $\bar{k} + 1$ is a power of two, so is $2\bar{k} + 1 + 1$ and $t_k = t_{2k+1} = 0$.

TABLE 1
Spanning Sets for Wavelet Packets

packet k	\bar{k}	Polynomials
0	0	$1, \cos x, \sin x, \cos 2x$
1	1	$\sin 2x, \cos 3x, \sin 3x, \cos 4x$
2	3	$\sin(6x - 1/4\pi), \cos 7x, \sin 7x, \cos 8x$
3	2	$\sin 4x, \cos 5x, \sin 5x, \cos(6x - 1/4\pi)$
4	7	$\sin(14x - 3/8\pi), \cos 15x, \sin 15x, \cos 16x$
5	6	$\sin(12x - 2/8\pi), \cos 13x, \sin 13x, \cos(14x - 3/8\pi)$
6	4	$\sin 8x, \cos 9x, \sin 9x, \cos(10x - 7/8\pi)$
7	5	$\sin(10x - 7/8\pi), \cos 11x, \sin 11x, \cos(12x - 2/8\pi)$
8	15	$\sin(30x - 7/16\pi), \cos 31x, \sin 31x, \cos 32x$
9	14	$\sin(28x - 6/16\pi), \cos 29x, \sin 29x, \cos(30x - 7/16\pi)$
10	12	$\sin(24x - 4/16\pi), \cos 25x, \sin 25x, \cos(26x - 1/16\pi)$
11	13	$\sin(26x - 1/16\pi), \cos 27x, \sin 27x, \cos(28x - 6/16\pi)$
12	8	$\sin 16x, \cos 17x, \sin 17x, \cos(18x - 11/16\pi)$
13	9	$\sin(18x - 11/16\pi), \cos 19x, \sin 19x, \cos(20x - 14/16\pi)$
14	11	$\sin(22x - 5/16\pi), \cos 23x, \sin 23x, \cos(24x - 4/16\pi)$
15	10	$\sin(20x - 14/16\pi), \cos 21x, \sin 21x, \cos(22x - 5/16\pi)$

Otherwise, $t_{2k+1} = t_k + (\bar{k} + 1)$, and $\cos((\bar{2k} + 1 + 1)2^{N-\ell}x - t_{2k+1}\pi) \in W_{N-\ell}^{(2k+1)}$, which concludes the proof for the case that \bar{k} is even.

As already mentioned, in the case where \bar{k} is odd, one uses essentially the same arguments with similar computations. ■

EXAMPLE 4.1. Consider the case $N = 5$, i.e.,

$$V_5 = \text{span}\{1, \cos x, \dots, \cos 31x, \cos 32x, \sin x, \dots, \sin 31x\}.$$

Choosing $\ell = 4$, one obtains 16 packets spanned by 4 functions each. For packet $k = 7$, $\epsilon^{(7)} = (0, 1, 1, 1)$ and thus $\bar{\epsilon}^{(7)} = (0, 1, 0, 1)$, yielding $\bar{k} = 5$, $u_k = 7/8$, and $t_k = 1/4$, so that $W_1^{(7)}$ is spanned by the functions $\sin(10x - 7/8\pi), \cos 11x, \sin 11x, \cos(12x - 1/4\pi)$. Altogether, one obtains Table 1.

We will now consider the reverse situation to that of Theorem 4.2, i.e., for a given trigonometric polynomial $\cos nx$ or $\sin nx$, $n = 1, \dots, 2^N - 1$, we seek the packet containing this function. Note beforehand that the constant function 1 is necessarily in packet 0, while $\cos 2^N x$ is in packet $2^{\ell-1}$ for any order ℓ .

For a dyadic vector $\underline{\epsilon}_\ell = (\epsilon_0, \dots, \epsilon_{\ell-1})$ of length ℓ , i.e., $\epsilon_r = 0$ or 1, we define another vector $\check{\underline{\epsilon}}_\ell = (\check{\epsilon}_0, \dots, \check{\epsilon}_{\ell-1})$ by

$$\check{\epsilon}_{\ell-1} := \epsilon_{\ell-1}$$

$$\check{\epsilon}_r := \begin{cases} \epsilon_r & \text{if } \epsilon_{r+1} = 0 \\ 1 - \epsilon_r & \text{if } \epsilon_{r+1} = 1 \end{cases} \quad \text{for } r = \ell - 2, \dots, 0. \quad (4.4)$$

A straightforward induction establishes that the operation $\check{}$ is actually the inverse to the operation $\bar{}$, defined in (4.3); i.e., for any dyadic vector, it holds that $\check{\check{\underline{\epsilon}}}_\ell = \underline{\epsilon}_\ell$. Con-

sequently, for a number $k = \sum_{r=1}^{\ell} \epsilon_{r-1} \cdot 2^{r-1}$,

$$\check{\bar{k}} = \sum_{r=1}^{\ell} \check{\epsilon}_{r-1} \cdot 2^{r-1} = k. \quad (4.5)$$

THEOREM 4.3. *For given $N \in \mathbb{N}_0$, the packet order $\ell = 1, \dots, N$, and a trigonometric polynomial $\cos nx$ or $\sin nx$, $n = 1, \dots, 2^N - 1$, consider a partial dyadic expansion of n modulo $2^{N-\ell}$, i.e.,*

$$n = 2^{N-\ell}n_\ell + \rho_\ell^{(n)},$$

with

$$n_\ell = \sum_{r=1}^{\ell} \eta_{r-1}^{(n)} \cdot 2^{r-1} \quad \text{and} \quad 0 \leq \rho_\ell^{(n)} \leq 2^{N-\ell} - 1.$$

Then it holds that

$$\sin nx \quad \text{and} \quad \cos nx \in W_{N-\ell}^{(\check{n}_\ell)} \quad \text{if } \rho_\ell^{(n)} \neq 0.$$

Otherwise, if $\rho_\ell^{(n)} = 0$ (meaning that n is divisible by $2^{N-\ell}$ without remainder), define $\check{n}_\ell = \check{}(n_\ell - 1)$, and we obtain

$$\sin(nx - u_{\check{n}_\ell}\pi) \in W_{N-\ell}^{(\check{n}_\ell)} \quad \text{and} \quad \cos(nx - t_{\check{n}_\ell}\pi) \in W_{N-\ell}^{(\check{n}_\ell)}$$

with $t_{\check{n}_\ell}$ and $u_{\check{n}_\ell}$ as defined above in Theorem 4.2.

Proof. According to Theorem 4.2, if $\rho_\ell^{(n)} \neq 0$, the space $W_{N-\ell}^{(\check{n}_\ell)}$ contains the function $\cos(\check{n}_\ell 2^{N-\ell} + \rho_\ell^{(n)})x = \cos nx$, as the operations $\check{}$ and $\bar{}$ are inverse to each other. Similarly, $\sin nx$ also is in $W_{N-\ell}^{(\check{n}_\ell)}$.

If $\rho_\ell^{(n)} = 0$, $W_{N-\ell}^{(\check{n}_\ell)}$ contains $\sin(\check{n}_\ell 2^{N-\ell} - u_{\check{n}_\ell}\pi) = \sin(nx - u_{\check{n}_\ell}\pi)$, while $W_{N-\ell}^{(\check{n}_\ell)}$ contains $\cos((\check{n}_\ell + 1)2^{N-\ell} - t_{\check{n}_\ell}\pi) = \cos(nx - t_{\check{n}_\ell}\pi)$. ■

EXAMPLE 4.2. We consider again the case $N = 5$ and $\ell = 4$. Suppose we would like to find the wavelet packets spaces containing the functions $\sin(18x - c_1), \cos(18x - c_1)$ and $\sin(27x - c_2), \cos(27x - c_2)$, where c_1 and c_2 are proper constants. Therefore, according to Theorem 4.3, we have to compute $18 = 2^{N-\ell}n_\ell + \rho_\ell^{(n)} = 2^{5-4} \cdot 9 + 0$, $\check{9} = 13$, and $27 = 2 \cdot 13 + 1$, $\check{13} = 11$. From $\rho_4^{(27)} = 1 \neq 0$ follows immediately that $c_2 = 0$ and $\sin(27x), \cos(27x)$ are contained in $W_1^{(11)}$. Since $\rho_4^{(18)} = 0$ we have to compute $\check{9} = \check{8} = 12$ and $t_{12} = u_{13} = 27\pi/16$, which tells us that $\sin(18 - 11\pi/16) \in W_1^{(13)}$ and $\cos(18 - 11\pi/16) \in W_1^{(12)}$.

5. MATRIX REPRESENTATIONS FOR WAVELET PACKETS

Analogous to Section 3, we now state the matrix formulation for the decomposition and reconstruction of the

coefficients of a function in the space V_{N+1} into/from the coefficients in its wavelet packet component spaces.

For a given $N \in \mathbb{N}_0$, we have

$$f_{N+1}^{(0)}(x) = f_N^{(0)}(x) + f_N^{(1)}(x),$$

$$\text{with } f_{N+1}^{(0)} \in W_{N+1}^{(0)}, f_N^{(0)} \in W_N^{(0)}, \text{ and } f_N^{(1)} \in W_N^{(1)}$$

and after another decomposition step:

$$f_N^{(0)}(x) = f_{N-1}^{(0)}(x) + f_{N-1}^{(1)}(x), \quad f_N^{(1)}(x) = f_{N-1}^{(2)}(x) + f_{N-1}^{(3)}(x),$$

with

$$f_{N-1}^{(i)} \in W_{N-1}^{(i)} \quad (i = 0, \dots, 2^2 - 1).$$

This yields

$$f_{N+1}^{(0)}(x) = f_{N-1}^{(0)}(x) + f_{N-1}^{(1)}(x) + f_{N-1}^{(2)}(x) + f_{N-1}^{(3)}(x).$$

Using the basis functions of these spaces, one obtains for $f_k^{(i)} \in W_k^{(i)}$

$$f_k^{(i)}(x) = \sum_{s=0}^{2^{k+1}-1} d_{k+1,s}^{(i)} \psi_{k,s}^{(i)}.$$

Hence after two decomposition steps it holds that

$$\begin{aligned} \sum_{s=0}^{2^{N+2}-1} d_{N+2,s}^{(0)} \psi_{N+1,s}^{(0)} &= \sum_{s=0}^{2^N-1} d_{N,s}^{(0)} \psi_{N-1,s}^{(0)} + \sum_{s=0}^{2^N-1} d_{N,s}^{(1)} \psi_{N-1,s}^{(1)} \\ &\quad + \sum_{s=0}^{2^N-1} d_{N,s}^{(2)} \psi_{N-1,s}^{(2)} + \sum_{s=0}^{2^N-1} d_{N,s}^{(3)} \psi_{N-1,s}^{(3)}. \end{aligned}$$

Now, define

$$\underline{d}_k^{(i)T} := (d_{k,0}^{(i)}, \dots, d_{k,2^k-1}^{(i)})$$

and

$$\underline{\psi}_k^{(i)T} := (\psi_{k-1,0}^{(i)}, \dots, \psi_{k-1,2^k-1}^{(i)})$$

$$\underline{d}_{N+2}^{(0)} = (\underline{d}_N^{(0)T}, \underline{d}_N^{(1)T}, \underline{d}_N^{(2)T}, \underline{d}_N^{(3)T}) \begin{pmatrix} \underline{\psi}_N^{(0)} \\ \underline{\psi}_N^{(1)} \\ \underline{\psi}_N^{(2)} \\ \underline{\psi}_N^{(3)} \end{pmatrix}.$$

Using the decomposition matrix from (2.10), two packet decomposition steps, as compared to (3.6), can now be written as

$$\begin{pmatrix} \underline{d}_N^{(0)} \\ \underline{d}_N^{(1)} \\ \underline{d}_N^{(2)} \\ \underline{d}_N^{(3)} \end{pmatrix} = \begin{pmatrix} D_{N+1}^T \Pi_{N+1} & 0 \\ 0 & D_{N+1}^T \Pi_{N+1} \end{pmatrix} D_{N+2}^T \Pi_{N+2} \underline{d}_{N+2}^{(0)}, \quad (5.1)$$

while in comparison to (3.5) two packet reconstruction steps are represented by

$$\Pi_{N+2} \underline{d}_{N+2}^{(0)} = R_{N+2}^T \begin{pmatrix} \Pi_{N+1}^T R_{N+1}^T & 0 \\ 0 & \Pi_{N+1}^T R_{N+1}^T \end{pmatrix} \begin{pmatrix} \underline{d}_N^{(0)} \\ \underline{d}_N^{(1)} \\ \underline{d}_N^{(2)} \\ \underline{d}_N^{(3)} \end{pmatrix}. \quad (5.2)$$

The factorized versions of these equations are for reconstruction

$$\begin{aligned} \underline{d}_{N+2}^{(0)} &= \Pi_{N+2}^T \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \begin{pmatrix} I_{2^{N+1}} & -\alpha_{N+1} \\ \bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \begin{pmatrix} \Pi_{N+1}^T & 0 \\ 0 & \Pi_{N+1}^T \end{pmatrix} \\ &\quad \times \begin{pmatrix} \bar{F}_N & 0 & 0 \\ 0 & \bar{F}_N & 0 \\ 0 & 0 & \bar{F}_N \end{pmatrix} \begin{pmatrix} I_{2^N} & -\alpha_N & 0 \\ \bar{\alpha}_N & I_{2^N} & 0 \\ 0 & 0 & I_{2^N} \end{pmatrix} \begin{pmatrix} F_N & 0 & 0 \\ 0 & F_N & 0 \\ 0 & 0 & F_N \end{pmatrix} \begin{pmatrix} \underline{d}_N^{(0)} \\ \underline{d}_N^{(1)} \\ \underline{d}_N^{(2)} \\ \underline{d}_N^{(3)} \end{pmatrix} \end{aligned}$$

and for decomposition

$$\begin{pmatrix} \underline{d}_N^{(0)} \\ \underline{d}_N^{(1)} \\ \underline{d}_N^{(2)} \\ \underline{d}_N^{(3)} \end{pmatrix} = \begin{pmatrix} \bar{F}_N & 0 & 0 & 0 \\ 0 & \bar{F}_N & 0 & 0 \\ 0 & 0 & \bar{F}_N & 0 \\ 0 & 0 & 0 & \bar{F}_N \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_N^{-1} & 0 & 0 & 0 \\ 0 & \tilde{\Gamma}_N^{-1} & 0 & 0 \\ 0 & 0 & \tilde{\Gamma}_N^{-1} & 0 \\ 0 & 0 & 0 & \tilde{\Gamma}_N^{-1} \end{pmatrix} \begin{pmatrix} I_{2^N} & \alpha_N & 0 & 0 \\ -\bar{\alpha}_N & I_{2^N} & 0 & 0 \\ 0 & 0 & I_{2^N} & \alpha_N \\ 0 & -\bar{\alpha}_N & I_{2^N} & 0 \end{pmatrix} \begin{pmatrix} F_N & 0 & 0 & 0 \\ 0 & F_N & 0 & 0 \\ 0 & 0 & F_N & 0 \\ 0 & 0 & 0 & F_N \end{pmatrix} \\ \times \begin{pmatrix} \Pi_{N+1} & 0 \\ 0 & \Pi_{N+1} \end{pmatrix} \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_{N+1}^{-1} & 0 \\ 0 & \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix} \begin{pmatrix} I_{2^{N+1}} & \alpha_{N+1} \\ -\bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \Pi_{N+2} \underline{d}_{N+2}^{(0)}.$$

In order to obtain fast decomposition and reconstruction algorithms, the transition matrices \tilde{T}_{N+2} and \tilde{U}_{N+2} of Section 3 are modified as follows:

$$\begin{aligned} \tilde{T}_{N+2} &:= \begin{pmatrix} T_{N+1} & 0 \\ 0 & T_{N+1} \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_{N+1}^{-1} & 0 \\ 0 & \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix} \\ &\quad \times \begin{pmatrix} I_{2^{N+1}} & \alpha_{N+1} \\ -\bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \\ &= \begin{pmatrix} T_{N+1} \tilde{\Gamma}_{N+1}^{-1} & T_{N+1} \tilde{\Gamma}_{N+1}^{-1} \alpha_{N+1} \\ -T_{N+1} \tilde{\Gamma}_{N+1}^{-1} \bar{\alpha}_{N+1} & T_{N+1} \tilde{\Gamma}_{N+1}^{-1} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \tilde{U}_{N+2} &:= \begin{pmatrix} I_{2^{N+1}} & -\alpha_{N+1} \\ \bar{\alpha}_{N+1} & I_{2^{N+1}} \end{pmatrix} \begin{pmatrix} U_{N+1} & 0 \\ 0 & U_{N+1} \end{pmatrix} \\ &= \begin{pmatrix} U_{N+1} & -\alpha_{N+1} U_{N+1} \\ \bar{\alpha}_{N+1} U_{N+1} & U_{N+1} \end{pmatrix}. \end{aligned}$$

The matrices T_{N+1} and U_{N+1} are defined in Section 3. Note that \tilde{T}_{N+2} and \tilde{U}_{N+2} are sparse matrices with 4 non-zero elements in each row. Explicit expressions for the transition matrices \tilde{T}_{N+2} and \tilde{U}_{N+2} are presented in

LEMMA 5.1. *For $N \in \mathbb{N}_0$, the transition matrices \tilde{T}_{N+2} and \tilde{U}_{N+2} each consist of 16 blocks of diagonal matrices of*

length 2^N . More precisely,

$$\tilde{T}_{N+2} := \frac{\sqrt{2}}{4} \begin{pmatrix} \begin{pmatrix} I_{2^N} & I_{2^N}^* \\ \bar{E}_N^{(2)} & -\bar{E}_N^{(2)*} \end{pmatrix} & \begin{pmatrix} E_N^{(1)} & -iE_N^{(1)*} \\ \bar{E}_N^{(1)} & i\bar{E}_N^{(1)*} \end{pmatrix} \\ \begin{pmatrix} -\bar{E}_N^{(1)} & -i\bar{E}_N^{(1)*} \\ -\bar{E}_N^{(3)} & i\bar{E}_N^{(3)*} \end{pmatrix} & \begin{pmatrix} I_{2^N} & I_{2^N}^* \\ \bar{E}_N^{(2)} & -\bar{E}_N^{(2)*} \end{pmatrix} \end{pmatrix}$$

and

$$\tilde{U}_{N+2} = \frac{\sqrt{2}}{2} \begin{pmatrix} \begin{pmatrix} I_{2^N} & E_N^{(2)} \\ I_{2^N} & -E_N^{(2)} \end{pmatrix} & \begin{pmatrix} -E_N^{(1)} & -E_N^{(3)} \\ iE_N^{(1)*} & -iE_N^{(3)*} \end{pmatrix} \\ \begin{pmatrix} \bar{E}_N^{(1)} & E_N^{(1)} \\ i\bar{E}_N^{(1)*} & -iE_N^{(1)*} \end{pmatrix} & \begin{pmatrix} I_{2^N} & E_N^{(2)} \\ I_{2^N} & -E_N^{(2)} \end{pmatrix} \end{pmatrix},$$

where $E_N^{(\ell)}, E_N^{(\ell)*} (\ell = 1, 2, 3)$, and $I_{2^N}^*$ are defined in Lemma 3.1.

Proof. The representations given in Lemma 3.1, formulae (2.11) and (2.12), and some straightforward computations yield the desired result. ■

Using these transition matrices reduces two reconstruction steps to

$$\underline{d}_{N+2}^{(0)} = \Pi_{N+2}^T \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \tilde{U}_{N+2} \begin{pmatrix} I_{2^N} & -\alpha_N & 0 \\ \bar{\alpha}_N & I_{2^N} & 0 \\ 0 & 0 & I_{2^N} & -\alpha_N \\ 0 & \bar{\alpha}_N & I_{2^N} & 0 \end{pmatrix} \begin{pmatrix} F_N & 0 & 0 \\ 0 & F_N & 0 \\ 0 & 0 & F_N \\ 0 & 0 & 0 & F_N \end{pmatrix} \begin{pmatrix} \underline{d}_N^{(0)} \\ \underline{d}_N^{(1)} \\ \underline{d}_N^{(2)} \\ \underline{d}_N^{(3)} \end{pmatrix}$$

and two decomposition steps to

$$\begin{pmatrix} \underline{d}_N^{(0)} \\ \underline{d}_N^{(1)} \\ \underline{d}_N^{(2)} \\ \underline{d}_N^{(3)} \end{pmatrix} = \begin{pmatrix} \bar{F}_N & 0 & 0 & 0 \\ 0 & \bar{F}_N & 0 & 0 \\ 0 & 0 & \bar{F}_N & 0 \\ 0 & 0 & 0 & \bar{F}_N \end{pmatrix} \begin{pmatrix} \tilde{\Gamma}_N^{-1} & 0 & 0 & 0 \\ 0 & \tilde{\Gamma}_N^{-1} & 0 & 0 \\ 0 & 0 & \tilde{\Gamma}_N^{-1} & 0 \\ 0 & 0 & 0 & \tilde{\Gamma}_N^{-1} \end{pmatrix} \begin{pmatrix} I_{2^N} & \alpha_N & 0 & 0 \\ -\bar{\alpha}_N & I_{2^N} & 0 & 0 \\ 0 & 0 & I_{2^N} & \alpha_N \\ 0 & -\bar{\alpha}_N & I_{2^N} & 0 \end{pmatrix} \tilde{T}_{N+2} \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \Pi_{N+2} \underline{d}_{N+2}^{(0)}.$$

Analogously, one obtains for ℓ reconstruction steps ($\ell \geq 2$):

$$\begin{aligned} \underline{d}_{N+2}^{(0)} &= \Pi_{N+2}^T \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \tilde{U}_{N+2} \begin{pmatrix} \tilde{U}_{N+1} & 0 \\ 0 & \tilde{U}_{N+1} \end{pmatrix} \cdots \begin{pmatrix} \tilde{U}_{N-\ell+4} & & & 0 \\ & \tilde{U}_{N-\ell+4} & & \\ & & \ddots & \\ 0 & & & \tilde{U}_{N-\ell+4} \\ & & & & \tilde{U}_{N-\ell+4} \end{pmatrix} \\ &\quad \times \begin{pmatrix} I_{2^{N-\ell+2}} & -\alpha_{N-\ell+2} & & 0 \\ \bar{\alpha}_{N-\ell+2} & I_{2^{N-\ell+2}} & & \\ & & \ddots & \\ 0 & & & I_{2^{N-\ell+2}} & -\alpha_{N-\ell+2} \\ & & \bar{\alpha}_{N-\ell+2} & I_{2^{N-\ell+2}} \end{pmatrix} \begin{pmatrix} F_{N-\ell+2} & & & 0 \\ & F_{N-\ell+2} & & \\ & & \ddots & \\ 0 & & & F_{N-\ell+2} \\ & & & & F_{N-\ell+2} \end{pmatrix} \begin{pmatrix} \underline{d}_{N-\ell+2}^{(0)} \\ \vdots \\ \underline{d}_{N-\ell+2}^{(2^\ell-1)} \end{pmatrix}, \end{aligned}$$

while ℓ decomposition steps ($\ell \geq 2$) are described by

$$\begin{aligned} \begin{pmatrix} \underline{d}_{N-\ell+2}^{(0)} \\ \vdots \\ \underline{d}_{N-\ell+2}^{(2^\ell-1)} \end{pmatrix} &= \begin{pmatrix} \bar{F}_{N-\ell+2} & & & 0 \\ & \bar{F}_{N-\ell+2} & & \\ & & \ddots & \\ 0 & & & \bar{F}_{N-\ell+2} \\ & & & & \bar{F}_{N-\ell+2} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \tilde{\Gamma}_{N-\ell+2}^{-1} & \tilde{\Gamma}_{N-\ell+2}^{-1} \alpha_{N-\ell+2} & & 0 \\ -\tilde{\Gamma}_{N-\ell+2}^{-1} \bar{\alpha}_{N-\ell+2} & \tilde{\Gamma}_{N-\ell+2}^{-1} & & \\ & & \ddots & \\ 0 & & & \tilde{\Gamma}_{N-\ell+2}^{-1} & \tilde{\Gamma}_{N-\ell+2}^{-1} \alpha_{N-\ell+2} \\ & & -\tilde{\Gamma}_{N-\ell+2}^{-1} \bar{\alpha}_{N-\ell+2} & \tilde{\Gamma}_{N-\ell+2}^{-1} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \tilde{T}_{N-\ell+4} & & & 0 \\ & \tilde{T}_{N-\ell+4} & & \\ & & \ddots & \\ 0 & & & \tilde{T}_{N-\ell+4} \\ & & & & \tilde{T}_{N-\ell+4} \end{pmatrix} \cdots \begin{pmatrix} \tilde{T}_{N+1} & 0 \\ 0 & \tilde{T}_{N+1} \end{pmatrix} \tilde{T}_{N+2} \begin{pmatrix} F_{N+1} & 0 \\ 0 & F_{N+1} \end{pmatrix} \Pi_{N+2} \underline{d}_{N+2}^{(0)}. \end{aligned}$$

Note that the first reconstruction step and the last decomposition step need to be treated differently; all others can be represented by transition matrices \tilde{T}_j and \tilde{U}_j of some order.

We now state the

ALGORITHM A3. Decomposition using semiorthogonal trigonometric wavelet packets

Input:

$N \in \mathbb{N}$ starting level (sufficiently large).
 \underline{d}_{N+2}^0 coefficients of the initial approximation $f_{N+1} \in V_{N+1}$ these coefficients are the 2^{N+2} function values taken at equidistant knots in the interval $[0, 2\pi)$.
 ℓ number of decomposition levels ($1 \leq \ell \leq N$).

(i) Reorder the elements of the vector \underline{d}_{N+2}^0 into a vector

$$\begin{pmatrix} \underline{d}_{N+1}^e \\ \underline{d}_{N+1}^o \end{pmatrix} := \Pi_{N+2} \underline{d}_{N+2}^0,$$

whereby $\underline{d}_{N+1}^e(\underline{d}_{N+1}^o)$ contains the elements of \underline{d}_{N+2}^0 with even (odd) indices.

(ii) Compute two Fourier transformations (of length 2^{N+1}) of the vectors \underline{d}_{N+1}^e and \underline{d}_{N+1}^o and set

$$\underline{\tilde{d}}_{N+2}^{(0)} := \begin{pmatrix} F_{N+1} \underline{d}_{N+1}^e \\ F_{N+1} \underline{d}_{N+1}^o \end{pmatrix}.$$

(iii) For each level $j(j = 0, \dots, \ell - 2)$ do
for each packet $k(k = 0, \dots, 2^j - 1)$ do

compute $\underline{\tilde{d}}_{N+2-j}^{(i)} := \tilde{T}_{N+2-j} \underline{\tilde{d}}_{N+2-j}^{(k)}$ and split

$$\underline{\tilde{d}}_{N+2-j}^{(i)} =: \begin{pmatrix} \underline{\tilde{d}}_{N+1-j}^{(2k)} \\ \underline{\tilde{d}}_{N+1-j}^{(2k+1)} \end{pmatrix}.$$

(iv) For each packet $k(k = 0, \dots, 2^{\ell-1} - 1)$ do
compute $\underline{\tilde{d}}_{N-\ell+3}^{(k)} := \tilde{T}_{N-\ell+3} \underline{\tilde{d}}_{N-\ell+3}^{(k)}$ and split

$$\underline{\tilde{d}}_{N-\ell+3}^{(k)} =: \begin{pmatrix} \underline{\tilde{d}}_{N-\ell+2}^{(2k)} \\ \underline{\tilde{d}}_{N-\ell+2}^{(2k+1)} \end{pmatrix}.$$

(v) For each packet $k(k = 0, \dots, 2^\ell - 1)$ do
compute one inverse Fourier transformation (of length $2^{N-\ell+2}$) of the vector $\underline{\tilde{d}}_{N-\ell+2}^{(k)}$, i.e., $\underline{d}_{N-\ell+2}^{(k)} := \bar{F}_{N-\ell+2} \underline{\tilde{d}}_{N-\ell+2}^{(k)}$.

Output: Vector of the wavelet packet coefficients $(\underline{d}_{N-\ell+2}^{(0)}, \underline{d}_{N-\ell+2}^{(1)}, \dots, \underline{d}_{N-\ell+2}^{(2^\ell-1)})^T$ and the corresponding

ALGORITHM A4. Reconstruction using semiorthogonal trigonometric wavelet packets.

Input:

$N \in \mathbb{N}$ starting level (determined by the decomposition algorithm A3).

$(\underline{d}_{N-\ell+2}^{(0)}, \underline{d}_{N-\ell+2}^{(1)}, \dots, \underline{d}_{N-\ell+2}^{(2^\ell-1)})^T$ wavelet packet coefficients from the decomposition algorithm A3.

ℓ number of reconstruction levels ($1 \leq \ell \leq N$).

(i) For each packet $k(k = 0, \dots, 2^\ell - 1)$ do
compute one Fourier transformation (of length $2^{N-\ell+2}$) of the vector $\underline{d}_{N-\ell+2}^{(k)}$, i.e., $\underline{\tilde{d}}_{N-\ell+2}^{(k)} := F_{N-\ell+2} \underline{d}_{N-\ell+2}^{(k)}$.

(ii) For each packet $k(k = 0, \dots, 2^{\ell-1} - 1)$ do
compute

$$\underline{\tilde{d}}_{N-\ell+3}^{(k)} =: \begin{pmatrix} \underline{\tilde{d}}_{N-\ell+2}^{(2k)} - \alpha_{N-\ell+2} \underline{\tilde{d}}_{N-\ell+2}^{(2k+1)} \\ -\bar{\alpha}_{N-\ell+2} \underline{\tilde{d}}_{N-\ell+2}^{(2k)} + \underline{\tilde{d}}_{N-\ell+2}^{(2k+1)} \end{pmatrix}.$$

(iii) For each level $j(j = \ell - 2, \dots, 0)$ do
for each packet $k(k = 0, \dots, 2^j - 1)$ do

compute

$$\underline{\tilde{d}}_{N+2-j}^{(k)} := \tilde{U}_{N+2-j} \begin{pmatrix} \underline{\tilde{d}}_{N+1-j}^{(2k)} \\ \underline{\tilde{d}}_{N+1-j}^{(2k+1)} \end{pmatrix}.$$

(iv) Compute two inverse Fourier transformations (of length 2^{N+1}) of the first 2^{N+1} elements of $\underline{\tilde{d}}_{N+2}^{(0)}$ and of the last 2^{N+1} elements of $\underline{\tilde{d}}_{N+2}^{(0)}$, i.e.,

$$\underline{d}_{N+2} =: \begin{pmatrix} \bar{F}_{N+1} & 0 \\ 0 & \bar{F}_{N+1} \end{pmatrix} \underline{\tilde{d}}_{N+2}^{(0)}.$$

(v) Reorder the elements of the vector \underline{d}_{N+2} into a vector denoted by \underline{d}_{N+2}^0 according to the rule

$$d_{N+2,i}^0 := \begin{cases} d_{N+2,i/2} & \text{if } i \text{ is even,} \\ d_{N+2,2^{N+1}+(i-1)/2} & \text{if } i \text{ is odd,} \end{cases} \quad (i = 0, \dots, 2^{N+2} - 1).$$

Output: Vector of the reconstructed coefficients \underline{d}_{N+2}^0 .

6. ORTHOGONAL WAVELETS AND WAVELET PACKETS

For certain applications which process the coefficients produced by the decomposition algorithm, for example, denoising with generalized cross validation, it is necessary to use not just semi-orthogonal bases for the wavelet and wavelet packet spaces, but truly orthonormal bases. As we have already computed the inner product matrices for scaling functions and wavelets, it is not very complicated to produce orthonormal bases and their decomposition and reconstruction matrices. Taking another look at the decomposition and reconstruction matrices in (2.7) and (2.10), we see that $R_{j+2} D_{j+2} = I_{2^{j+2}}$ means

$$\begin{pmatrix} I_{2^{j+1}} & K_{j+1} \\ -K_{j+1}^T & I_{2^{j+1}} \end{pmatrix} \begin{pmatrix} \frac{1}{2^{j+2}} G_{j+1}^{-1} & 0 \\ 0 & \frac{1}{2^{j+2}} G_{j+1}^{-1} \end{pmatrix} \begin{pmatrix} I_{2^{j+1}} & -K_{j+1} \\ K_{j+1}^T & I_{2^{j+1}} \end{pmatrix} = I_{2^{j+2}}.$$

To create orthogonal decomposition and reconstruction matrices, it seems appropriate to just spread the inverse of the inner product matrix more evenly among decomposition and reconstruction by writing

$$\begin{pmatrix} I_{2^{j+1}} & K_{j+1} \\ -K_{j+1}^T & I_{2^{j+1}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2^{j+2}}} G_{j+1}^{-1/2} & 0 \\ 0 & \frac{1}{\sqrt{2^{j+2}}} G_{j+1}^{-1/2} \end{pmatrix} \\ \times \begin{pmatrix} \frac{1}{\sqrt{2^{j+2}}} G_{j+1}^{-1/2} & 0 \\ 0 & \frac{1}{\sqrt{2^{j+2}}} G_{j+1}^{-1/2} \end{pmatrix} \begin{pmatrix} I_{2^{j+1}} & -K_{j+1} \\ K_{j+1}^T & I_{2^{j+1}} \end{pmatrix} = I_{2^{j+2}},$$

where $G_{j+1}^{-1/2}$ is the well-defined square root matrix of the symmetric positive definite matrix G_{j+1}^{-1} . It turns out that the orthogonalization procedure described in more detail below indeed produces the reconstruction matrix

$$R_{j+2}^\perp = \begin{pmatrix} I_{2^{j+1}} & K_{j+1} \\ -K_{j+1}^T & I_{2^{j+1}} \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2^{j+2}}} G_{j+1}^{-1/2} & 0 \\ 0 & \frac{1}{\sqrt{2^{j+2}}} G_{j+1}^{-1/2} \end{pmatrix} \quad (6.1)$$

and the decomposition matrix

$$D_{j+2}^\perp = \begin{pmatrix} \frac{1}{\sqrt{2^{j+2}}} G_{j+1}^{-1/2} & 0 \\ 0 & \frac{1}{\sqrt{2^{j+2}}} G_{j+1}^{-1/2} \end{pmatrix} \begin{pmatrix} I_{2^{j+1}} & -K_{j+1} \\ K_{j+1}^T & I_{2^{j+1}} \end{pmatrix}. \quad (6.2)$$

As the factorizations of the inner product matrices G_{j+1} and H_{j+1} and their inverses are already known from Theorems 2.2 and 2.3, it is easy to derive the matrices $G_{j+1}^{-1/2}$ and $H_{j+1}^{-1/2}$. In fact, we have

LEMMA 6.1 [22]. *For given $j \in \mathbb{N}_0$, it holds that*

$$G_{j+1}^{-1/2} = \sqrt{2^{j+1}} \bar{F}_{j+1} \text{diag}(1, \dots, 1, \underbrace{\sqrt{2}}_{2^j}, 1, \dots, 1) F_{j+1}$$

and thus

$$G_{j+1}^{-1/2} = \sqrt{2^{j+1}} I_{2^{j+1}} + \frac{\sqrt{2}-1}{\sqrt{2^{j+1}}} \chi_{j+1}.$$

Similarly,

$$H_{j+1}^{-1/2} = \sqrt{2^{j+1}} \bar{F}_{j+1} \text{diag}(\underbrace{\sqrt{2}}_1, 1, \dots, 1, \underbrace{\sqrt{2}}_{2^j}, 1, \dots, 1) F_{j+1}$$

or

$$\begin{aligned} H_{j+1}^{-1/2} &= \sqrt{2^{j+1}} I_{2^{j+1}} + \frac{\sqrt{2}-1}{\sqrt{2^{j+1}}} 1_{j+1} + \frac{\sqrt{2}-1}{\sqrt{2^{j+1}}} \chi_{j+1} \\ &= G_{j+1}^{-1/2} + \frac{\sqrt{2}-1}{\sqrt{2^{j+1}}} 1_{j+1}. \end{aligned}$$

We now introduce a new set of scaling functions and wavelets in

DEFINITION 6.1. For given $j \in \mathbb{N}_0$, a set of functions in V_j and W_j , respectively, is defined by

$$(\underline{\phi}_{j+1,n}^\perp)_{n=0,\dots,2^{j+1}-1} = G_{j+1}^{-1/2} (\underline{\phi}_{j+1,n})_{n=0,\dots,2^{j+1}-1}$$

and

$$(\underline{\psi}_{j+1,n}^\perp)_{n=0,\dots,2^{j+1}-1} = H_{j+1}^{-1/2} (\underline{\psi}_{j+1,n})_{n=0,\dots,2^{j+1}-1}.$$

One easily verifies the following

THEOREM 6.1 [22]. *For given $j \in \mathbb{N}_0$, the sets of functions $\phi_{j,k}^\perp$ and $\psi_{j,k}^\perp$ form an orthonormal basis of V_j and an orthonormal basis of W_j , respectively.*

By multiplying the reconstruction equation (2.6) for the original basis by the matrix

$$\begin{pmatrix} G_{j+1}^{-1/2} & 0 \\ 0 & H_{j+1}^{-1/2} \end{pmatrix},$$

one obtains

$$\begin{aligned} \begin{pmatrix} \underline{\phi}_{j+1}^\perp \\ \underline{\psi}_{j+1}^\perp \end{pmatrix} &= \begin{pmatrix} G_{j+1}^{-1/2} & 0 \\ 0 & H_{j+1}^{-1/2} \end{pmatrix} \begin{pmatrix} \underline{\phi}_{j+1} \\ \underline{\psi}_{j+1} \end{pmatrix} \\ &= \begin{pmatrix} G_{j+1}^{-1/2} & 0 \\ 0 & H_{j+1}^{-1/2} \end{pmatrix} R_{j+2} \Pi_{j+2} \underline{\phi}_{j+2} \\ &= \begin{pmatrix} G_{j+1}^{-1/2} & 0 \\ 0 & H_{j+1}^{-1/2} \end{pmatrix} \\ &\quad \times R_{j+2} \Pi_{j+2} G_{j+2}^{1/2} \Pi_{j+2}^T \Pi_{j+2} G_{j+2}^{-1/2} \underline{\phi}_{j+2} \\ &= \begin{pmatrix} G_{j+1}^{-1/2} & 0 \\ 0 & H_{j+1}^{-1/2} \end{pmatrix} \\ &\quad \times R_{j+2} \Pi_{j+2} G_{j+2}^{1/2} \Pi_{j+2}^T \Pi_{j+2} \underline{\phi}_{j+2}^\perp. \end{aligned}$$

Using that

$$G_{j+2}^{1/2} = \frac{1}{\sqrt{2^{j+2}}} I_{2^{j+2}} + \left(\frac{1}{\sqrt{2}} - 1 \right) \frac{1}{\sqrt{2^{3(j+2)}}} \chi_{j+2},$$

there remain some lengthy, but feasible calculations to establish that indeed

$$R_{j+2}^\perp = \begin{pmatrix} G_{j+1}^{-1/2} & 0 \\ 0 & H_{j+1}^{-1/2} \end{pmatrix} R_{j+2} \Pi_{j+2} G_{j+2}^{1/2} \Pi_{j+2}^T$$

for the matrix R_{j+2}^\perp introduced in (6.1):

TABLE 2

Wavelet Decomposition: Computation Time in Seconds

	Number of Data					
	2048	4096	8192	16384	32768	65536
otw deco	0.33	0.51	1.04	2.59	5.5	11.85
otw reco	0.29	0.47	1.02	2.53	5.4	11.5
db8 deco	0.06	0.12	0.24	0.48	0.97	1.96
db8 reco	0.05	0.11	0.22	0.44	0.88	1.77
db18 deco	0.13	0.26	0.51	1.01	2.02	4.06
db18 reco	0.10	0.21	0.42	0.85	1.71	3.40
cubspl deco	0.04	0.08	0.16	0.33	0.66	1.32
cubspl reco	0.02	0.03	0.07	0.16	0.32	0.64

THEOREM 6.2. *For given $j \in \mathbb{N}_0$, and the orthonormal bases $\phi_{j,k}^\perp$ and $\psi_{j,k}^\perp$, the reconstruction matrix is R_{j+2}^\perp of (6.1) and the decomposition matrix is D_{j+2}^\perp of (6.2).*

For the algorithms described in Sections 3 and 5, all there is to do is to replace the eigenvalues in the factorizations by the ones of the orthogonal matrices. The functions investigated here are based on the Fourier means and turn out to be quite close to orthonormal, i.e., after a proper normalization only one eigenvalue in the factorization of the inner product matrix of scaling functions is different from one, and only two are different for the wavelet matrix. This means that the effect of orthonormalization in the factorization of the decomposition and reconstruction matrices is limited to only the few entries where the eigenvalue different from one is involved. This must then be also true for the wavelet packet constructions whose functions turn out to be orthogonal, too, as orthogonal matrices are now used in their construction. For example, in Lemma 5.1, apart from the factors for \tilde{U}_{N+2} and \tilde{T}_{N+2} , only the two leading entries of $E_N^{(0)*}$ and $E_N^{(2)*}$ are changed from 2 to $\sqrt{2}$. For higher de la Vallée–Poussin-type summations, the orthogonalization affects more entries, as more eigenvalues of the inner product matrices are different from one. Essentially, though, all algorithms can be adapted for those means as well.

7. NUMERICAL APPLICATIONS

7.1. Computational effort and stability

We start this section with some remarks about the complexity and stability of the decomposition algorithms A1 and A3. Note that the reconstruction algorithms A2 and A4 have about the same complexity as the respective decomposition algorithms. Only the essential operations are counted, i.e., real and complex multiplications.

Suppose we want to decompose 2^N given data values into $\ell > 1$ levels using ordinary wavelets (not wavelet packets). Disregarding the normalization factors in the matrices \tilde{T}_N

and \hat{T}_N , this involves the following numbers of operations for algorithm A1:

- 2 fast Fourier transformations (FFT's) of length 2^{N-1} to compute step (ii),
 - on each level $j(j = 0, \dots, \ell - 2)$
 - 4 complex multiplications for each of the first 2^{N-j-1} coefficients,
 - 2 complex multiplications for each of the second 2^{N-j-1} coefficients,
 - 1 FFT of length 2^{N-1-j} to compute step (iii),
 - 2 complex multiplications for each of $2^{N-\ell+1}$ coefficients to compute step (iv),
 - 2 FFT's of length $2^{N-\ell}$ to compute step (v).
- In total, this amounts to

$$2 \cdot O((N-1)2^{N-1}) + 2 \cdot O((N-\ell)2^{N-\ell}) + \sum_{j=0}^{\ell-2} O((N-j-1)2^{N-j-1})$$

real multiplications to perform the FFT's and $3 \sum_{j=0}^{\ell-2} 2^{N-j} + 2 \cdot 2^{N-\ell+1} = 6 \cdot 2^N - 2 \cdot 2^{N-\ell+2}$ complex multiplications to perform the ℓ decomposition steps in the Fourier domain, where $O(N \cdot 2^N)$ is the number of real multiplications needed to perform an FFT of length 2^N . The number of operations can be further reduced because of the special structure of the matrices \tilde{T}_N and \hat{T}_N (some submatrices are identity matrices). Our implementation of the decomposition in the Fourier domain needs

$$7 \sum_{j=0}^{\ell-2} 2^{N-j} + 6 \cdot 2^{N-\ell+1} = 14 \cdot 2^N - 4 \cdot 2^{N-\ell+2}$$

real multiplications and the computation of $\sum_{j=0}^{\ell-2} 2^{N-j} = 2^{N+1} - 2^{N-\ell+2}$ cosine and sine values, respectively. In comparison, a straightforward implementation of the decomposition algorithm using compactly supported orthogonal wavelets with M filter coefficients (e.g., Daubechies wavelets) needs $M \sum_{j=0}^{\ell-1} 2^{N-j} = 2 \cdot M(2^N - 2^{N-\ell})$ real multiplications.

Table 2 shows the computation times for one level decomposition and reconstruction on a SPARCstation2

TABLE 3

Wavelet Packet Decomposition: Computation Time in Seconds

	Number of Data					
	2048	4096	8192	16384	32768	65536
otw deco	0.68	1.32	2.94	6.72	14.91	32.65
otw reco	0.81	1.80	4.06	9.27	20.26	44.07
db8 deco	0.48	1.07	2.56	5.40	11.80	25.21
db8 reco	0.41	0.93	2.09	4.62	10.18	22.06
db18 deco	1.05	2.36	5.03	11.08	24.46	53.18
db18 reco	0.84	1.87	4.39	9.45	20.38	44.20
cubspl deco	0.28	0.61	1.36	2.95	6.21	13.56
cubspl reco	0.11	0.27	0.63	1.50	3.09	6.81

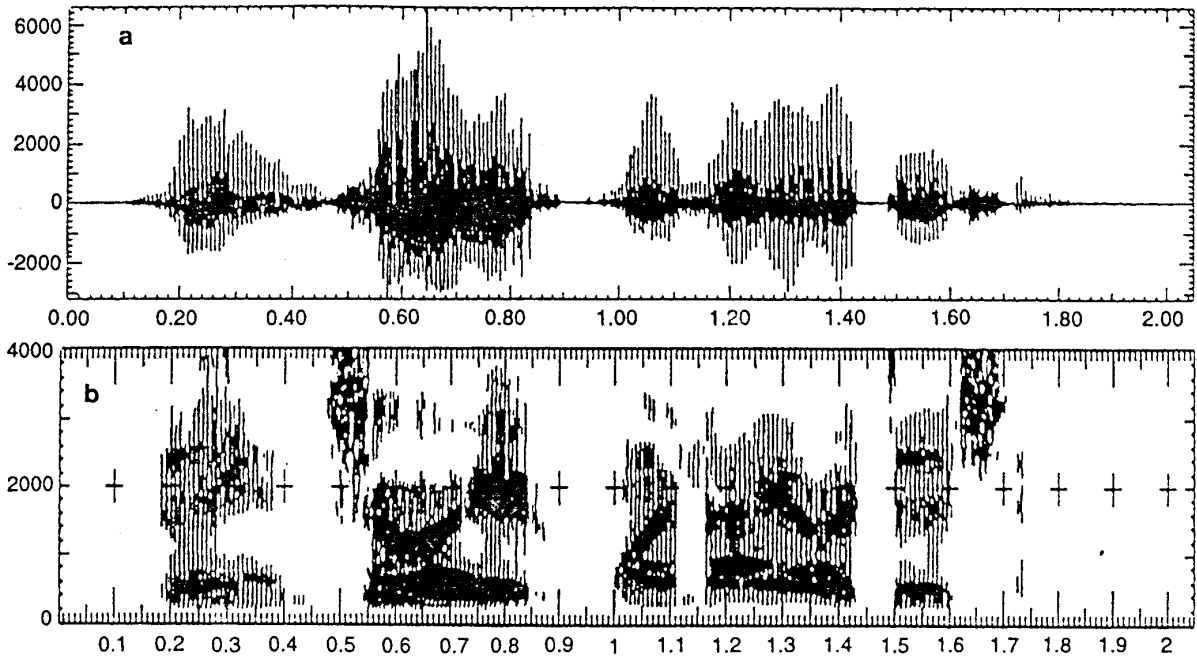


FIGURE 1

for orthogonal trigonometric wavelets (otw), periodized Daubechies wavelets with 8 filter coefficients (db8), and 18 filter coefficients (db18) (see [12]) and boundary corrected cubic spline wavelets (cubspl) using a fast algorithm described in [25]. Since most of the computation time in Algorithm A1 is used to perform the FFT's, an increase of

the number of decomposition steps would only have a small effect on the computation time.

Note that our implementation of the decomposition and reconstruction algorithm for periodized Daubechies wavelets does not use a fast algorithm, because the length of the filter coefficients is not fixed (i.e., it can be chosen

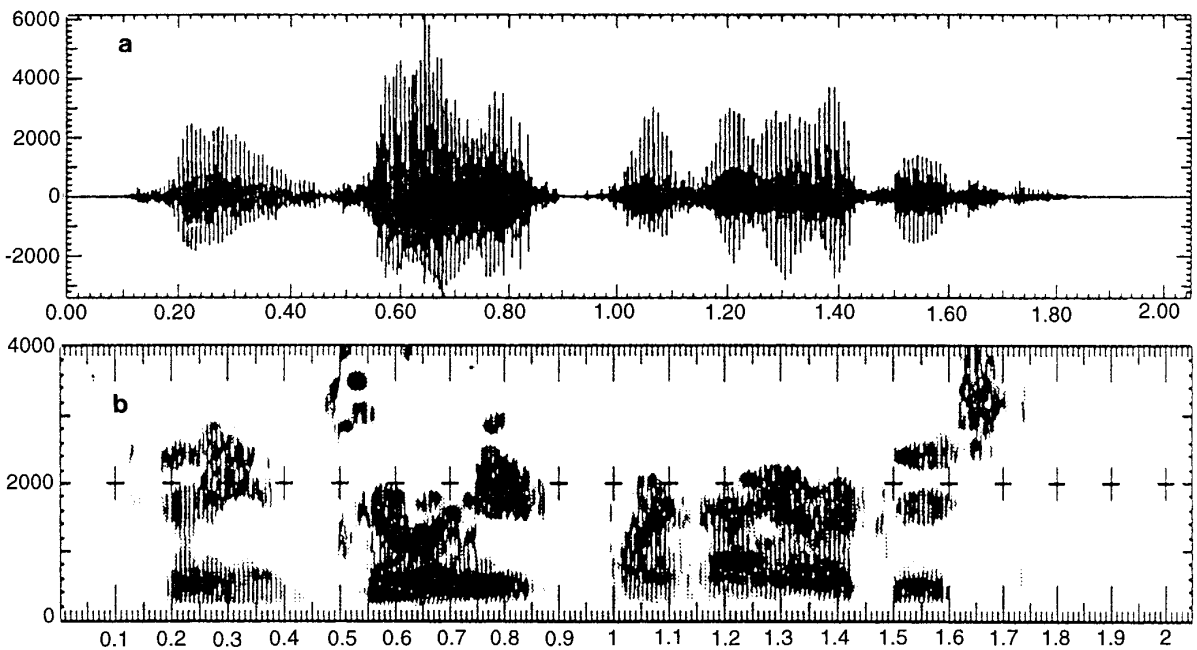


FIGURE 2

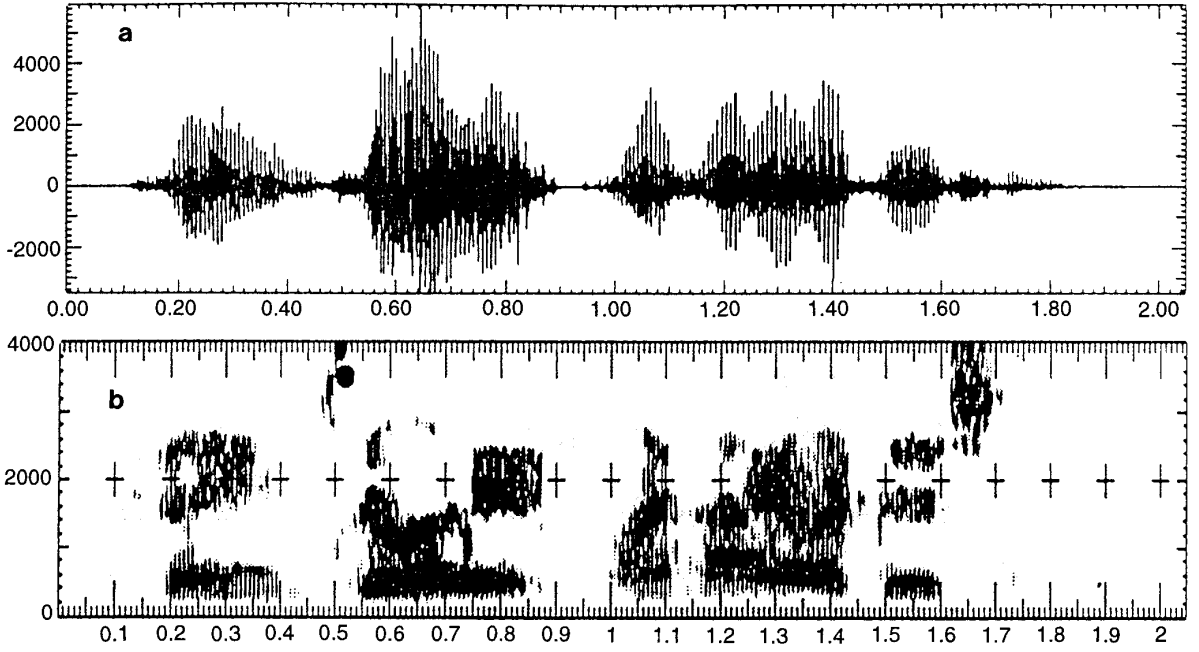


FIGURE 3

by setting a parameter). This fact, together with the periodization which therefore necessitates costly if-tests, slows down the algorithm significantly.

In order to decompose 2^N data values into $\ell > 1$ levels using the wavelet packet algorithm A3 we need (disregarding again the normalization factors in the matrices \tilde{T}_N and \hat{T}_N)

- 2 FFT's of length 2^{N-1} to compute step (ii),
- on each level $j (j = 0, \dots, \ell - 2)$ $2^j \cdot 4 \cdot 2^{N-j}$ complex multiplications to compute step (iii),
- $2^{\ell-1} \cdot 2 \cdot 2^{N-\ell+1}$ complex multiplications to compute step (iv)
- 2^ℓ FFT's of length $2^{N-\ell}$ to compute step (v), i.e., $2 \cdot O((N-1)2^{N-1}) + 2^\ell \cdot O((N-\ell)2^{N-\ell})$ real multiplications.

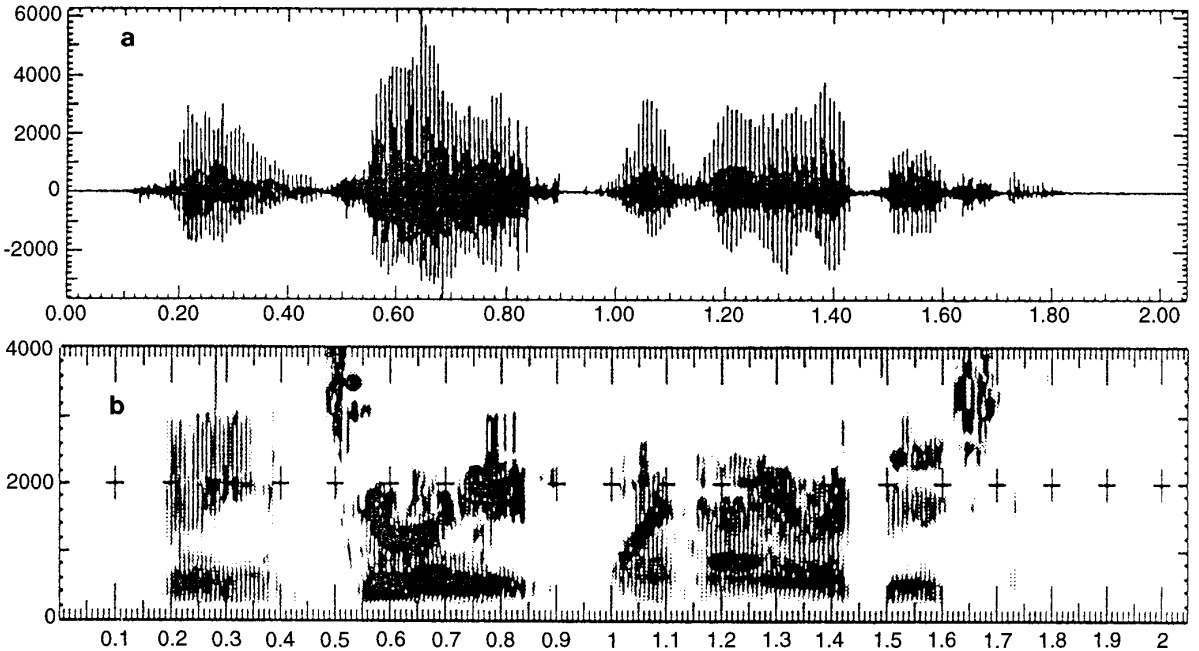


FIGURE 4

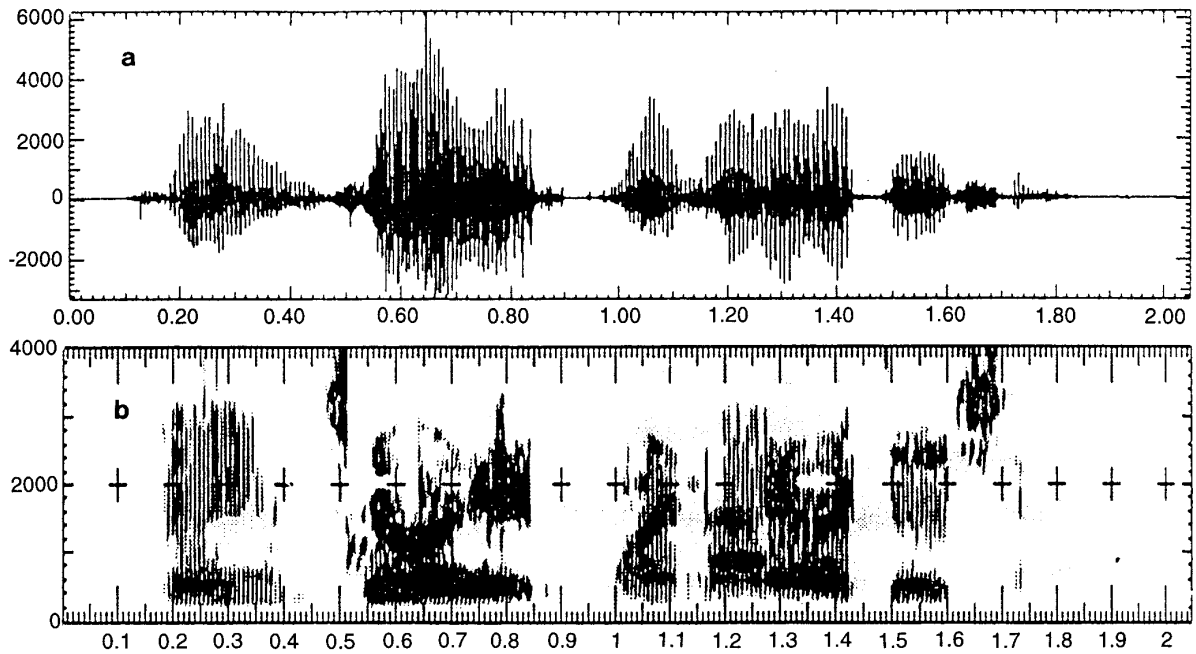


FIGURE 5

tions to perform the FFT's and $4(\ell - 1)2^N + 2 \cdot 2^N$ complex multiplications for the ℓ decomposition steps in the Fourier domain (steps (iii) and (iv) of Algorithm A3). For the decomposition in the Fourier domain our implementation needs $8(\ell - 1)2^N + 6 \cdot 2^N$ real multiplications and the computation of $3 \sum_{j=0}^{\ell-1} 2^{N-j-1} = 3(2^N - 2^{N-\ell})$ cosine and sine values, respectively. Again we compare these numbers with a decomposition using compactly supported orthogonal wavelets with filter length M . Here we need $M \cdot \ell \cdot 2^N$ real multiplications for a wavelet packet decomposition. In Table 3, the computation times are shown for an $N = 3$ level wavelet packet decomposition, with 2^N given data values.

Note that in Table 2 only the computation time for the trigonometric wavelets is increasing by a factor of $N \cdot 2^N$ (due to the FFT's involved), while the increase is linear for Daubechies wavelets and spline wavelets. Quite differently, in Table 3 the order of increase (approximately $O(N \cdot 2^N)$) is the same for all tested wavelet types.

It also turns out that the decomposition and reconstruction algorithms A1–A4 are stable. While the FFT matrices are unitary anyway, the condition numbers for the (different) transition matrices \tilde{T}_{N+2} of Lemmas 3.1 and 5.1 and their inverses \tilde{U}_{N+2} can be computed explicitly, using the matrix norm $\|A\|_2 = \rho(\tilde{A}^T A)$, with ρ denoting the spectral radius. A straightforward computation shows that the matrices $\tilde{T}_{N+2}^T \tilde{T}_{N+2}$ have an eigenvalue 1 of multiplicity 2 and an eigenvalue $1/2$ of multiplicity $2^{N+2} - 2$, while the matrices $\tilde{U}_{N+2}^T \tilde{U}_{N+2}$ have an eigenvalue 1 of multiplicity 2 and

an eigenvalue 2 of multiplicity $2^{N+2} - 2$. Consequently, the condition numbers of the transition matrices are always 2, regardless of the level N .

7.2. Applications in Speech Processing

Since trigonometric wavelets consist of sine and cosine functions, it seems rather natural to apply those wavelets to audio signals. In this section, an application of trigonometric wavelets to speech compression is given. Another application of trigonometric wavelets to speech denoising using generalized cross validation can be found in [28]. Besides the discrete wavelet transforms, both applications use the thresholding technique, as described in [14, 27]. We also compare our results with those achieved using Daubechies wavelets.

We assume that data $y_i = y(t_i)$ are given for $t_i = i/n$ ($i = 1, \dots, n$). In our setup the number of given data values will always be $n = 2^N$. The *discrete wavelet transform* $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)^T$ of a given data vector $\underline{y} = (y_1, \dots, y_n)^T$ is defined by the equation $\hat{y} := W\underline{y}$, and the *inverse discrete wavelet transform* by $\underline{y} := W^{-1}\hat{y}$, where the $n \times n$ matrix $W = W(\ell)$ performs the wavelet and the wavelet packet decomposition into ℓ levels, respectively, and $W^{-1} = W^{-1}(\ell)$ performs the wavelet and wavelet packet reconstruction, respectively. For notational convenience we will suppress the index ℓ as well as the dimension n . Explicit expressions for the matrices W for both the wavelet and the wavelet packet cases can be found in [27]. For orthogonal wavelets, the matrix W is orthogonal and we have $W^{-1} = W^T$.

For further improvement of the results we use the notion of best basis described in [11]. As in [11], we re-

strict the set of all possible bases to those provided by a wavelet packet decomposition. This results in a binary tree and makes it easy to search for an optimal basis. As the cost function for a vector, we use the entropy defined as $\lambda(x) := -\sum_{j=1}^m x_j^2 \log(x_j^2)$ ($x \in \mathbb{R}^m$). The best basis is then defined as that collection of nonoverlapping basis vectors, provided by a wavelet packet decomposition, which has minimal entropy. In this case the matrix W performs a *best basis decomposition* and W^{-1} the respective *best basis reconstruction*.

Furthermore, define for a fixed *threshold parameter* δ , the *thresholded* vectors $\hat{y}_\delta = (\hat{y}_{\delta_1}, \dots, \hat{y}_{\delta_n})^T$ by $\hat{y}_\delta := D_\delta \hat{y}$ and $y_\delta = (y_{\delta_1}, \dots, y_{\delta_n})^T$ by $y_\delta := W^{-1} \hat{y}_\delta$, with $D_\delta = D_\delta(y) := \text{diag}(1, \dots, 1, d_{\delta_{m_\ell+1}}, \dots, d_{\delta_n})$, where

$$d_{\delta_i} := \begin{cases} 0 & \text{if } |\hat{y}_i| \leq \delta, \\ \left(1 - \frac{\lambda \delta}{|\hat{y}_i|}\right) & \text{if } |\hat{y}_i| > \delta, \end{cases} \quad (i = m_\ell + 1, \dots, n),$$

and $m_\ell := 2^{N-\ell}$. The parameter $\lambda \in [0, 1]$ determines the thresholding method. We will use $\lambda = 0$ (*ordinary* or *hard thresholding*) for speech compression. Another common choice, $\lambda = 1$ (*soft thresholding* or *shrinkage*), is widely used for the denoising problem (see [14, 27]). The vector \hat{y}_δ will be considered as the “compressed” signal, where in this case compression means simply increasing the number of zeros. The threshold parameter δ is chosen so that the number of nonzero elements in \hat{y}_δ is a certain percentage (10% in our example) of the total number of elements in $\hat{y}_\delta (= 2^N)$.

Our speech example is the sentence “They enjoy it when I audition,” spoken by a male voice, from the TIMIT data base. The total number of given data is 16384 and the sampling frequency is 8 kHz. In a first step, this sentence is segmented (or windowed) into 16 pieces which may not necessarily have the same length. The purpose of the segmentation is to distinguish between silence, unvoiced and voiced parts of the speech. However, in order to avoid using preinformation about the speech, a uniform segmentation with 1024 data in each window is used. The speech compression method is then performed on each segment.

Figures 1 and 2 (top, time domain; bottom, frequency domain) show the original speech and the compressed speech, respectively. The compressed speech y_δ is obtained by reconstructing the hard thresholded coefficients, the threshold parameter δ being chosen so that 90% of the wavelet coefficients \hat{y} in each window are zero. For the discrete wavelet transform we used trigonometric wavelet packets (Fig. 2) and, in comparison, periodized Daubechies wavelet packets with 20 filter coefficients (db20) in Fig. 3 (top, time domain; bottom, frequency domain). In this example, both wavelet types produced about the same (acceptable) sound quality. The relative error $\|\underline{y} - \underline{y}_\delta\|_2^2 / \|\underline{y}\|_2^2$ is 0.071 (or 7.7%) in the

trigonometric wavelet case and 10.35% in the Daubechies wavelet case. An improvement in both sound quality and the relative error was obtained by using a best basis rather than the pure wavelet packet basis. Figures 4 and 5 (top, time domain; bottom, frequency domain) show the compressed signal using the best basis for trigonometric wavelets and Daubechies wavelets, respectively. In the best basis case the relative error is 5.5% for trigonometric wavelets and 6.5% for Daubechies wavelets. Note that the use of Daubechies wavelets in combination with a filter process always introduces artificial frequency content in the filtered signal. This is a result of the (unwanted) side loops of the frequency spectrum while the frequency spectrum for trigonometric wavelets does not have any such side loops (see [22]). In order to minimize these effects Daubechies wavelets with a large number of filter coefficients must be used.

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